Quiz 10 solution.

Problem.
Let $\mathbf{r}(t) = (3t, 2t^2)$. **Find:**
(a) $\mathbf{r}'(t)$
(b) $\mathbf{r}''(t)$
For $t = 1$ find:
(c) $\vec{v}$
(d) $\vec{a}$
(e) $\hat{T}$
(f) ds/dt
(g) d^2s/dt^2
(h) a_T
(i) $\vec{a}_T$
(j) a_N
(k) $\vec{a}_N$
(l) K
where:
$\vec{v} =$ *velocity*
$\vec{a} =$ *acceleration*
$s =$ *distance along curve*
$\hat{T} =$ *unit tangent*
$\vec{a}_T =$ *projection of* $\vec{a}$ *onto* $\hat{T}$
$\hat{N} =$ *unit normal*
$\vec{a}_N =$ *projection of* $\vec{a}$ *onto* $\hat{N}$

Basic identities.
$a = \vec{a}_T + \vec{a}_N\hat{N}$, i.e.
$a = a_T\hat{T} + a_N\hat{N}$, i.e.
$a = \frac{d^2s}{dt^2}\hat{T} + (\frac{ds}{dt})^2 K\hat{N}$
$|v| = \frac{ds}{dt}$, $a_T = \frac{d^2s}{dt^2}$, and $K = \frac{a_N}{|v|^2}$.

Solution.

Raw derivatives.

(a) $\mathbf{r}'(t) = (3, 4t) = 3\hat{\imath} + 4t\hat{\jmath}$

(b) $\mathbf{r}''(t) = (0, 4) = 4\hat{\jmath}$

(c) $\vec{v} = \mathbf{r}'(1) = (3, 4) = 3\hat{\imath} + 4\hat{\jmath}$

(d) $\vec{a} = \mathbf{r}''(1) = (0, 4) = 4\hat{\jmath}$

First-derivative stuff.

(e) $|\vec{v}| = \sqrt{3^2 + 4^2} = 5$
So $\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{5}(3, 4) = \frac{3}{5}\hat{\imath} + \frac{4}{5}\hat{\jmath}$

(f) $\frac{ds}{dt}|_{t=1} = |\vec{v}| = 5$.
or compute directly using:
$\frac{ds}{dt}(t) = |\mathbf{r}'(t)| = \sqrt{3^2 + (4t)^2}$,
which equals 5 when $t = 1$.

Second-derivative tangential stuff.

(g) $\frac{d^2s}{dt^2} = a_T = \vec{a} \cdot \hat{T} = (0, 4) \cdot \frac{1}{5}(3, 4) = \frac{16}{5}$
or compute directly using:
$\frac{d}{dt}\frac{ds}{dt}(t) = 4\frac{4t}{\sqrt{3^2+(4t)^2}}$,
which equals $\frac{16}{5}$ when $t = 1$.

(h) $a_T = \frac{d^2s}{dt^2} = \frac{16}{5}$
as computed in part (g).

(i) $\vec{a}_T = a_T\hat{T} = (\frac{16}{5})(\frac{1}{5}(3, 4)) = \frac{4^2}{5^2}(3, 4) = (\frac{48}{25}, \frac{64}{25})$.

Second-derivative perpendicular stuff.

(j) $a_N^2 = |\vec{a}|^2 - a_T^2 = 4^2 - (\frac{16}{25})^2 5^2 = 4^2 - \frac{4^2 4^2}{5^2} = \frac{4^2 5^2 - 4^2 4^2}{5^2} = \frac{4^2 \cdot 3^2}{5^2}$. So $a_N = \frac{4 \cdot 3}{5} = \frac{12}{5}$.
Or just do part (k) and use $a_N = |\vec{a}_N|$.

(k) $\vec{a}_N = \vec{a} - \vec{a}_T = (0, 4) - \frac{1}{25}(48, 64)$
$= \frac{1}{25}(-48, 100 - 64) = \frac{1}{25}(-48, 36)$

(l) $K = \frac{a_N}{|\vec{v}|^2} = \frac{12/5}{25} = \frac{12}{125}$
Or use the formula:
$K = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{3/2}} = \frac{|3 \cdot 4 - 4t \cdot 0|}{(3^2 + (4t)^2)^{3/2}} = \frac{12}{(3^2 + (4t)^2)^{3/2}}$,
which equals $\frac{12}{125}$ when $t = 1$.