Convergence Tests for Series

Background.
We say that A **converges absolutely** if $\sum_{k=k_0}^{\infty} |a_k|$ converges. If A converges absolutely, then A converges. If A converges but does not converge absolutely, then we say that A **converges conditionally**. If A does not converge (to a finite number), then we say that A **diverges**.

Context.
Let $A = \sum a_n := \sum_{n=n_0}^{\infty} a_n$.
Let $B = \sum_{n=n_0}^{\infty} b_n$.
Let $f(n) = a_n$. (f is a function which agrees with the sequence $(a_n)_{n=1}^{\infty}$.)
Let $I = \int_{x_0}^{\infty} f(x)\,dx$.

Convergence Test		**Assumptions**		**Conclusion (Does A converge?)**		
Book ref.	**Test name**	**abs.**	**other conditions**	**converges**	**diverges**	**inconclusive**
§10.2 p437	limit of terms	no	$(a_n \nrightarrow 0)$	N/A	if $a_n \nrightarrow 0$	if $a_n \to 0$
§10.3 p444	integral comp.	yes	f cont., $+$, $\searrow$	A iff I	($\bar{A}$ iff $\bar{I}$)	never
§10.4 p448	ordinary comp.	yes	$0 \le a_k \le b_k$	A if B	($\bar{B}$ if $\bar{A}$)	otherwise
§10.4 p449	limit comp.	yes	$\frac{a_n}{b_n} \to L \neq 0$:	A iff B	($\bar{A}$ iff $\bar{B}$)	$-$
			$\frac{a_n}{b_n} \to L = 0$:	A if B	($\bar{B}$ if $\bar{A}$)	otherwise
			$\frac{a_n}{b_n} \to \infty$:	(B if A)	$\bar{A}$ if $\bar{B}$	otherwise
§10.5 p455	ratio	yes	$\lvert\frac{a_{n+1}}{a_n}\rvert \to \rho$	if $\rho < 1$	if $\rho > 1$	if $\rho = 1$
(not in book)	n-th root test	yes	$\sqrt[n]{a_n} \to \rho$	if $\rho < 1$	if $\rho > 1$	if $\rho = 1$
§10.5 p454	alternating series	NO	alternating, $\lvert a_n\rvert \searrow$	if $a_n \to 0$	if $a_n \nrightarrow 0$	$-$

$f \searrow$ means that f is a *nonincreasing* function ($f(x) \ge f(y)$ if $x < y$).
(Parenthesized statements can be inferred from the other information in the table.)
The column labeled **abs.** indicates whether the test assumes that the terms are positive (in which case the test can only be used to show *absolute* convergence).

Test name	**When used**
limit of terms	Used to show divergence. Can't use to show convergence.
integral comparison test	Usually used to show absolute convergence when the formula for the terms of the series is a function you can integrate.
ordinary comparison	Used to show absolute convergence (respectively divergence) by comparing with a simpler series whose terms are larger (respectively smaller).
limit comparison	Used to show absolute convergence or diverence by comparing with a simpler series (e.g. a p-series).
ratio and n-th root	Used to show absolute convergence. Used most often for series with powers and factorials (e.g. to find the radius of convergence of a power series). Can't use for a function that involves only addition, subtraction, multiplication, division, roots, or (constant) powers, because it will always be inconclusive.
alternating series	Used most often to show convergence at an endpoint of the interval of convergence of a power series.

Example of Series		**When does it converge?**		
Name	**Series**	**converges**	**diverges**	**inconclusive**
p-series	$\sum_{n=0}^{\infty} \frac{1}{n^p}$	if $p > 1$	if $p \le 1$	
geometric series	$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$	if $\lvert x\rvert < 1$	if $\lvert x\rvert \ge 1$	
power series	$\sum_{n=0}^{\infty} c_n x^n$	if $\lvert x\rvert < \rho$	if $\lvert x\rvert > \rho$	if $\lvert x\rvert = \rho$
collapsing sum	$\sum_{n=0}^{\infty}(a_{n+1} - a_n) = -a_0$	if $a_n \to 0$	if $a_n \nrightarrow 0$	