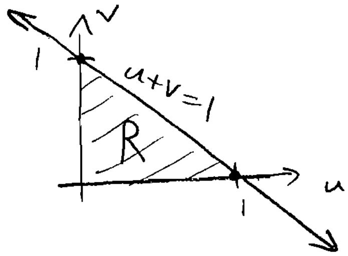
Section 15.1

(15) $f(u,v) := v - \sqrt{u}$



$$I = \iint_R f(u,v)\,du\,dv$$

$$= \int_{u=0}^{1} \int_{v=0}^{1-u} v - \sqrt{u}\,dv\,du$$

$$= \int_{u=0}^{1} \left[\frac{v^2}{2} - \sqrt{u}\,v\right]_{v=0}^{1-u} du$$

$$= \int_{u=0}^{1} \frac{(1-u)^2}{2} - \sqrt{u}\,(1-u)\,du$$

$$= \int_{u=0}^{1} \frac{1-2u+u^2}{2} - u^{\frac{1}{2}} + u^{3/2}\,du$$

$$= \left[\frac{u-u^2}{2} + \frac{u^3}{6} - \frac{2}{3}u^{3/2} + \frac{2}{5}u^{\frac{5}{2}}\right]_{u=0}^{1}$$

$$= \left[\frac{1}{6} - \frac{2}{3} + \frac{2}{5}\right]$$

$$= \frac{5-20+12}{30}$$

$$= \frac{-3}{30}$$

$$= \boxed{\frac{-1}{10}}$$

OR

$$I = \int_{v=0}^{1} \int_{u=0}^{1-v} v - \sqrt{u}\,du\,dv$$

$$= \int_{v=0}^{1} \left[vu - \frac{2}{3}u^{3/2}\right]_{u=0}^{1-v} dv$$

$$= \int_{v=0}^{1} v(1-v) - \frac{2}{3}(1-v)^{3/2}\,dv$$

[Let $w = 1-v$, so $dw = -dv$

$$= -\int_{w=1}^{0} (1-w)\,w - \frac{2}{3}w^{3/2}\,dw$$

$$= \int_{w=0}^{1} w - w^2 - \frac{2}{3}w^{3/2}\,dw$$

$$= \left[\frac{w^2}{2} - \frac{w^3}{3} - \frac{2}{3}\cdot\frac{2}{5}w^{5/2}\right]_{w=0}^{1}$$

$$= \left[\frac{1}{2} - \frac{1}{3} - \frac{4}{15}\right]$$

$$= \frac{15-10-8}{30}$$

$$= \frac{-3}{30} = \boxed{\frac{-1}{10}}$$

OR

$$I = \int_{v=0}^{1} \int_{u=0}^{1-v} v\,du\,dv - \int_{u=0}^{1} \int_{v=0}^{1-u} \sqrt{u}\,dv\,du$$

$$= \int_{v=0}^{1} \left[uv\right]_{u=0}^{1-v} dv - \int_{u=0}^{1} \left[\sqrt{u}\,v\right]_{v=0}^{1-u} du$$

$$= \int_0^1 v(1-v)\,dv - \int_0^1 \sqrt{u}\,(1-u)\,du$$

$$= \left[\frac{v^2}{2} - \frac{v^3}{3}\right]_0^1 - \left[\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right]_0^1$$

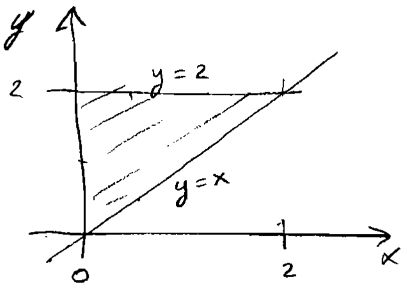
$$= \frac{1}{2} - \frac{1}{3} \quad - \frac{2}{3} + \frac{2}{5}$$

$$= \frac{2}{5} - \frac{1}{2} = \frac{4}{10} - \frac{5}{10} = \boxed{\frac{-1}{10}}$$

§15.1

(32) $\boxed{\int_0^2 \int_{y=x}^{2}} 2y^2 \sin xy \; dy\, dx$

$$\left\{ \begin{matrix} 0 \le x \le 2 \\ x \le y \le 2 \end{matrix} \right\} \Leftrightarrow \left\{ \begin{matrix} 0 \le y \le 2 \\ 0 \le x \le y \end{matrix} \right\}$$



$$= \int_{y=0}^{2} \int_{x=0}^{y} 2y^2 \sin xy \; dx\, dy$$

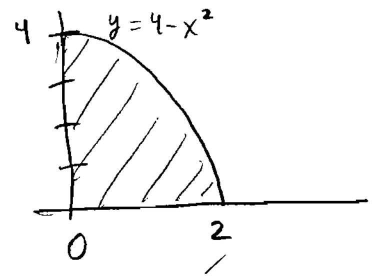
$$= -\int_{y=0}^{2} 2y^2 \left[\frac{1}{y} \cos(xy) \right]_{x=0}^{y} dy$$

$$= -\int_{y=0}^{2} 2y \left[\cos(y^2) - 1 \right] dy$$

$$= \left[-\sin(y^2) + y^2 \right]_{y=0}^{2}$$

$$= \boxed{4 - \sin(4)}$$

(34) $\displaystyle\int_{x=0}^{2} \int_{y=0}^{4-x^2} \frac{x e^{2y}}{4-y} \, dy\, dx$



$$= \int_{y=0}^{4} \int_{x=0}^{\sqrt{4-y}} \frac{x e^{2y}}{4-y} \, dx\, dy$$

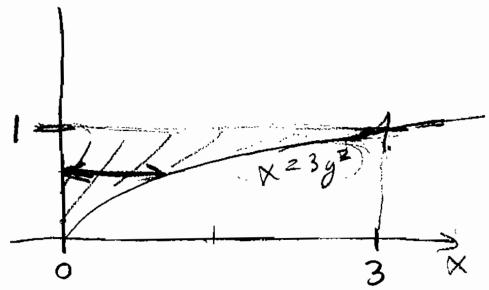
$$= \int_{y=0}^{4} \frac{e^{2y}}{4-y} \left[\frac{4-y}{2} \right] dy$$

$$= \left[\frac{e^{2y}}{4} \right]_{y=0}^{4}$$

$$= \boxed{\frac{e^8 - 1}{4}}$$

§15.1

(36) $$\int_{x=0}^{3}\int_{y=\sqrt{x/3}}^{1} e^{y^3}\,dy\,dx$$



$$= \int_{y=0}^{1}\int_{0}^{3y^2} e^{y^3}\,dx\,dy$$

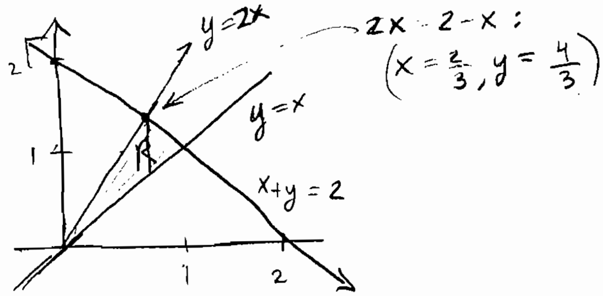
$$= \int_0^1 e^{y^3}\int_0^{3y^2} dx\,dy$$

$$= \int_0^1 3y^2 e^{y^3}\,dy$$

$$= \left[e^{y^3}\right]_{y=0}^{1}$$

$$= \boxed{e^1 - 1}$$

$\circledast$ (40)



$$\iint_R xy\,dA$$

$$= \int_{x=0}^{2/3}\int_{y=x}^{2x}(xy)\,dy\,dx + \int_{x=2/3}^{1}\int_{y=x}^{2-x}(xy)\,dy\,dx$$

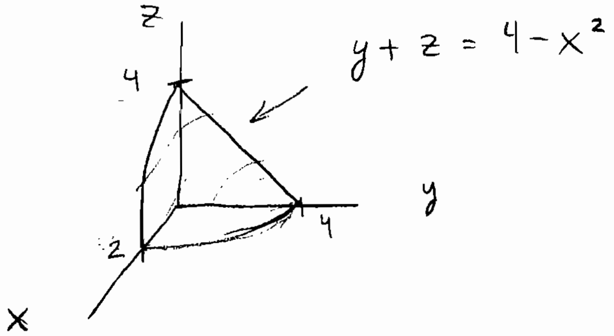
$$= \frac{1}{2}\int_{x=0}^{2/3} x\left[3x^2\right]dx + \frac{1}{2}\int_{x=2/3}^{1} x\left[(2-x)^2 - x^2\right]dx$$

$$= \frac{3}{8}\left[x^4\right]_{x=0}^{2/3} + \int_{x=2/3}^{1}(2x-2x^2)\,dx$$

$$= \frac{2}{27} + \left[x^2 - \frac{2}{3}x^3\right]_{x=2/3}^{1} = \underbrace{\frac{2}{27} + \frac{1}{3} - \frac{4}{9}}_{-\frac{1}{27}} + \frac{16}{81} = \boxed{\frac{13}{81}}$$

§15.1

(46)



$$V = \int_{x=0}^{2} \int_{y=0}^{4-x^2} \int_{z=0}^{4-x^2-y} dz\,dy\,dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{4-x^2} (4-x^2-y)\,dy\,dx$$

$$= \int_{x=0}^{2} (4-x^2)(4-x^2) - \left[\frac{y^2}{2}\right]_0^{4-x^2} dx$$

$$= \frac{1}{2}\int_{x=0}^{2} (4-x^2)^2\,dx$$

$$= \frac{1}{2}\int_{x=0}^{2} 16 - 8x^2 + x^4\,dx$$

$$= \frac{1}{2}\left[16x - \frac{8}{3}x^3 + \frac{x^5}{5}\right]_{x=0}^{2}$$

$$= \frac{1}{2}\left[2^5 - \frac{2^6}{3} + \frac{2^5}{5}\right]$$

$$= 2^4\left[1 - \frac{2}{3} + \frac{1}{5}\right]$$

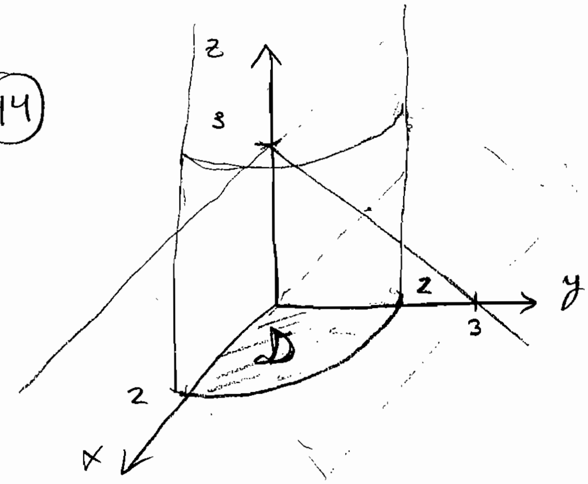
$$= 2^4 \cdot \left(\frac{1}{3} + \frac{1}{5}\right)$$

$$= 2^4 \cdot \frac{8}{15}$$

$$= \frac{2^7}{15}$$

$$= \boxed{\frac{128}{15}}$$

(44)



Find the volume V of the solid in the first octant bounded by the surfaces $x^2+y^2=4$ (a cylinder) and $z+y=3$ (a plane).

Solution

$$V = \iint_{D} \int_{z=0}^{3-y} dz\,dA$$

$$= \iint_{D} (3-y)\,dA$$

$$= \iint_{D} 3\,dA - \iint_{D} y\,dA$$

$$= 3\pi - \int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} y\,dy\,dx$$

$$= 3\pi - \int_{x=0}^{2} \frac{4-x^2}{2}\,dx$$

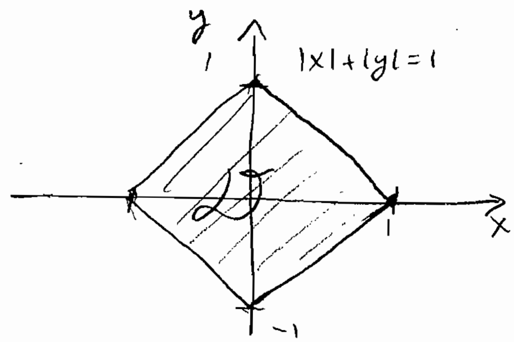
$$= 3\pi - \frac{1}{2}\left[4x - \frac{x^3}{3}\right]_{x=0}^{2}$$

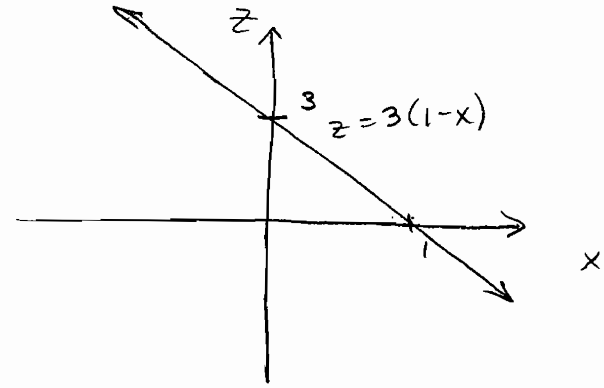
$$= 3\pi - \frac{1}{2}\left[8 - \frac{8}{3}\right]$$

$$= 3\pi - \frac{1}{2}\left(\frac{16}{3}\right)$$

$$= \boxed{3\pi - \frac{8}{3}}$$

(48) Find volume in cylinder $|x|+|y|=1$ between $z=0$ and $z=3-3x$.





$$V = \iint_{\mathcal{D}} \int_{z=0}^{3(1-x)} dz$$

$$= \iint_{\mathcal{D}} 3(1-x)\, dz$$

$$= 3\int_{x=-1}^{1} (1-x) \int_{y=-(1-|x|)}^{1-|x|} dy \quad dx$$

$$= 3\int_{x=-1}^{1} (1-x)\, 2(1-|x|)\, dx$$

$$= 6\int_{x=-1}^{1} (1-x)(1-(\operatorname{sgn} x)\, x)\, dx$$

$$= 6\int_{x=-1}^{1} 1-(1+\operatorname{sgn} x)\, x + \underbrace{(\operatorname{sgn} x)\, x^2}_{\text{odd}} dx$$

$$= 6\left[\int_{-1}^{1} 1\, dx - \int_{x=0}^{1} 2x\, dx\right]$$

$$= 6[2-1]$$

$$= \boxed{6}$$

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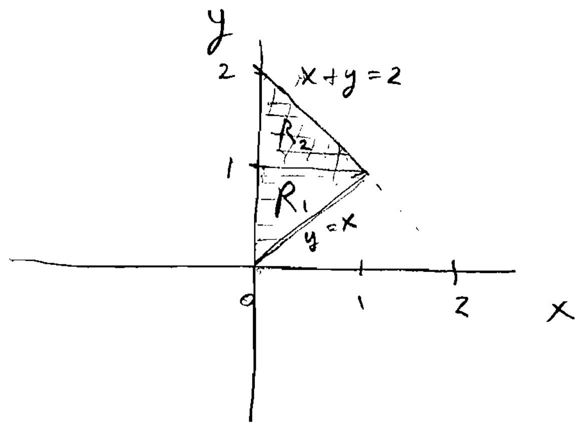


$$V=\underbrace{\int_{y=0}^{1}\int_{x=0}^{y}}_{R_1}(x^2+y^2)\,dx\,dy+\underbrace{\int_{y=1}^{2}\int_{x=0}^{2-y}}_{R_2}(x^2+y^2)\,dx\,dy$$

$$R=R_1\cup R_2$$

$$V=\int_{x=0}^{1}\int_{y=x}^{2-x}(x^2+y^2)\,dy\,dx$$

$$=\int_{x=0}^{1}\left[x^2y+\frac{y^3}{3}\right]_{y=x}^{2-x}dx$$

$$=\int_{x=0}^{1}x^2(2-2x)+\frac{(2-x)^3-x^3}{3}\,dx$$

$$=\int_{x=0}^{1}2x^2-2x^3+\frac{8-12x+6x^2-2x^3}{3}\,dx$$

$$=\int_{x=0}^{1}\frac{8-12x+12x^2-8x^3}{3}\,dx$$

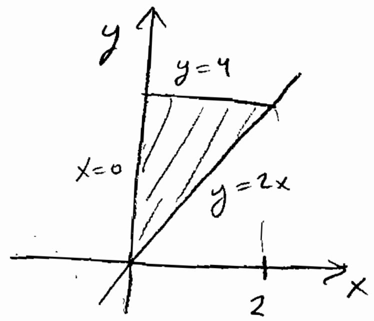
$$=\frac{1}{3}\left[8x-6x^2+4x^3-2x^4\right]_{x=0}^{1}$$

$$=\frac{1}{3}(8-6+4-2)$$

$$=\boxed{\frac{4}{3}}$$

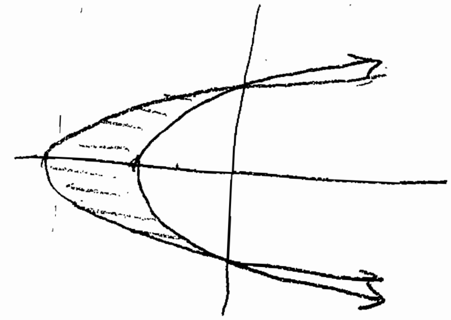
§15.2

$\boxed{2}$



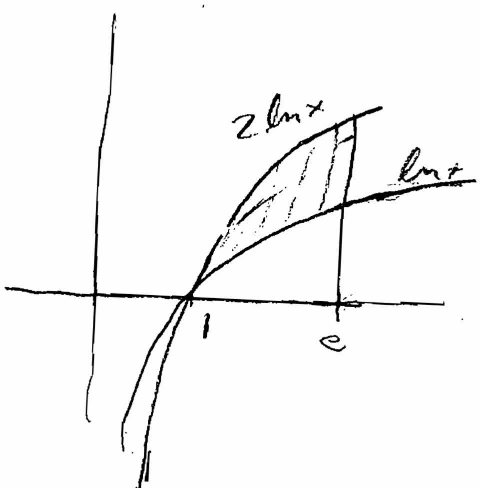
$$\text{area} = \int_{x=0}^{2}\int_{y=2x}^{4} dy\,dx$$
$$= \int_{x=0}^{2}(4-2x)\,dx$$
$$= 2\cdot 4 - [x^2]_0^2$$
$$= 8-4$$
$$= \boxed{4}$$

$\boxed{8}$



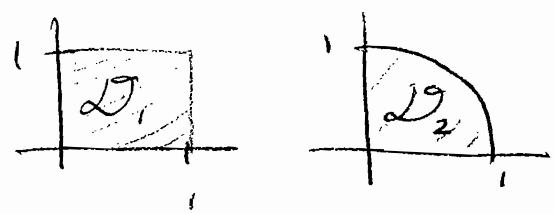
$$\text{Area} = \int_{y=-1}^{1}\int_{x=2(y^2-1)}^{(y^2-1)} dx\,dy$$
$$= \int_{y=-1}^{1} 1-y^2\,dy$$
$$= \left[y-\frac{y^3}{3}\right]_{-1}^{1}$$
$$= 2\cdot\left[1-\frac{1}{3}\right] = \boxed{\frac{4}{3}}$$

$\boxed{6}$



$$\text{Area} = \int_{x=1}^{e}\int_{y=\ln x}^{2\ln x} dy\,dx$$
$$= \int_{x=1}^{e}\ln x\,dx$$
$$= \left[x\ln x - x\right]_{x=1}^{e}$$
$$= e-(e-1)$$
$$= \boxed{1}$$

$\boxed{16}$ $f(x,y) = xy$



$$\fint\fint_{\mathcal{D}_1} f = \int_0^1\int_0^1 xy\,dx\,dy$$
$$= \left[\frac{x^2}{2}\right]_{x=0}^{1}\cdot\left[\frac{y^2}{2}\right]_{y=0}^{1}$$
$$= \boxed{\frac{1}{4}}$$

$$\fint\fint_{\mathcal{D}_2} xy = \frac{\iint_{\mathcal{D}_2} xy\,dx\,dy}{(\pi/4)} \leftarrow \iint_{\mathcal{D}_2} dx\,dy$$

$$\iint_{\mathcal{D}_1} xy\,dx\,dy = \int_{x=0}^{1} x\int_{y=0}^{\sqrt{1-x^2}} y\,dy\,dx$$
$$= \int_{x=0}^{1} x\cdot\frac{1}{2}(1-x^2)\,dx$$
$$= \frac{1}{2}\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 = \boxed{\frac{1}{8}}$$

OR: $$\iint_{\mathcal{D}_1} xy\,dA = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^2\cos\theta\sin\theta\,r\,dr\,d\theta$$
$$= \left(\int_{\theta=0}^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)d\theta\right)\left(\int_{r=0}^{1} r^3\,dr\right) = \left[\frac{\cos 2\theta}{4}\right]_{\frac{\pi}{2}}^{0}\cdot\left[\frac{r^4}{4}\right]_0^1 = \frac{1}{8}$$

$$\therefore \fint\fint_{\mathcal{D}_2} xy = \frac{1}{8}\cdot\frac{4}{\pi} = \frac{1}{2\pi} < \frac{1}{6} < \frac{1}{4}$$