Extra Credit due Wed Oct 29

p1064

(3) 3 Leibniz's Rule.

Suppose:

f continuous on $[a,b]$,

u, v are differentiable and have values in $[a,b]$.

Show:

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

Suggestion:

Let $g(u,v) = \int_u^v f(t)\,dt$.

Proof

$$LHS = \frac{d}{dx} g(u(x), v(x))$$

$$= \frac{\partial g}{\partial u}\cdot\frac{du}{dx} + \frac{\partial g}{\partial v}\frac{dv}{dx}$$

$$= -f(u(x))\frac{du}{dx} + f(v(x))\frac{dv}{dx}$$

$$= RHS.$$

(8) 4 [Radially symmetric function with zero Laplacian].

Suppose

$f: \mathbb{R}\to\mathbb{R}$ is twice differentiable,

$\tilde{f}(\vec{r}) = f(r)$ satisfies Laplace's equation,

$$\tilde{f}_{xx} + \tilde{f}_{yy} + \tilde{f}_{zz} = 0,$$

where $\vec{r} = (x,y,z)$

and $r = \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$

Show

$\exists$ constants a, b such that

$$f(r) = \frac{a}{r} + b.$$

Proof

We use the chain rule to express the Laplacian in terms of the derivatives of f:

$$\tilde{f}_x = f'(r)\cdot r_x.$$

$$\tilde{f}_{xx} = f''(r)\,r_x^2 + f'(r)\,r_{xx}.$$

$$r_x = \frac{x}{r}, \quad r_{xx} = \frac{r - \frac{x^2}{r}}{r^2} = \frac{r^2-x^2}{r^3}.$$

So

$$\tilde{f}_{xx} = \left(\frac{x}{r}\right)^2 f'' + \frac{r^2-x^2}{r^3} f'$$

$$\tilde{f}_{yy} = \left(\frac{y}{r}\right)^2 f'' + \frac{r^2-y^2}{r^3} f'$$

$$\tilde{f}_{zz} = \left(\frac{z}{r}\right)^2 f'' + \frac{r^2-z^2}{r^3} f'$$

(add)

$$0 = \left(\frac{r^2}{r^2}\right) f'' + \frac{3r^2-r^2}{r^3} f',$$

(3) i.e. $0 = f'' + \frac{2}{r} f'$

This is a 1st order homogeneous ODE in $g := f'$:

$$0 = g' + \frac{2}{r} g.$$ Consider $g \neq 0$. (else $f = \frac{0}{r} + b$)

Then $\frac{g'}{g} = -\frac{2}{r}$.

So $\ln|g| = \ln r^{-2} + C$, some C.

So $|f'| = |g| = (e^C)\, r^{-2}$

So $f = \pm(e^C)\cdot r^{-1} + b$, i.e.

(5) $$\boxed{f = \frac{a}{r} + b}$$ (check it!) $\Rightarrow f' = \frac{-a}{r^2},\ f'' = \frac{2a}{r^3}$ ✓

(5) 5 Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree n:

(*) $f(t\vec{r}_0) = f(\vec{r}_0)\ \forall t>0, \forall \vec{r}_0$.

Show:

(2) (a) $\vec{r}_0\cdot\nabla f(\vec{r}_0) = n f(\vec{r}_0)\ \forall \vec{r}_0$,

(3) (b) $(\vec{r}_0\cdot\nabla)^2 f(\vec{r}_0) = n(n-1) f(\vec{r}_0)$.

Proof

For (a), differentiate (*) with respect to t and evaluate at $t=1$: $\left[r_0\cdot\nabla f(t r_0) = n t^{n-1} f(\vec{r}_0)\right]\Big|_{t=1}$.

For (b) differentiate (*) twice with respect to t (again using the chain rule) and evaluate at $t=1$:

$$\left[(r_0\cdot\nabla)^2 f(t\vec{r}_0) = n(n-1)t^{n-2} f(\vec{r}_0)\right]\Big|_{t=1}$$