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1. **Warm-up.** Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in the plane.

   (a) What is the set of all vectors $s\,\mathbf{u} + t\,\mathbf{v}$ for $0 \leq s \leq 1$ and $0 \leq t \leq 1$?

   (b) What is the set of all vectors $\mathbf{u} + t\,\mathbf{v}$ for $t \in (-\infty, \infty)$?

   In order to get started, it might be useful to pick a few values for s and t and graph the resulting vectors.

2. **Curvature review.**

   (a) What is $\mathbf{T}(t)$ and why is it called the unit tangent vector?

   (b) What is $s(t)$, what does $\frac{d\mathbf{T}}{ds}$ represent, and why is it called the curvature vector?

   (c) Why does κ equal $\frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$ and what does it represent?

3. **Radius of curvature.**

   Let $\mathbf{r}(t) = a\cos(t)\mathbf{i} + a\sin(t)\mathbf{j}$.

   (a) Find the velocity, speed, and unit tangent at time t.

   (b) Find the tangential ($\mathbf{a}_T$) and normal ($\mathbf{a}_N$) components of the acceleration vector $\mathbf{a}$ at time t. Draw a graph illustrating your answers when $t = \pi/6$.

   (c) Find the curvature and the radius of curvature at time t.

   (d) How would your answers change if you chose a different value for t?

4. **Curvature of a spiral.**

   Let $\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t \rangle$.

   (a) Sketch the curve $\mathbf{r}(t)$ for $t \geq 0$.

   (b) Find the unit tangent vector $\mathbf{T}(t)$ at time t and sketch it when $t = \pi/3$. Does your answer agree with your sketch?

   (c) Calculate the acceleration vector $\mathbf{a}$ at time $t = \pi/3$. Compute the normal and tangential components of the acceleration at time $t = \pi/3$. Verify that your answers are correct by showing that $\|\mathbf{a}\|^2 = {a_T}^2 + {a_N}^2$ at $t = \pi/3$.

   (d) Estimate the curvature κ at time $t = \pi/3$ based on your sketch. Then calculate it.

   (e) Find the equation of the *osculating circle* tangent to the curve.

5. **Kiss my curve.**

   Consider $\mathbf{r}(t) = t\sin t\mathbf{i} + 3t\mathbf{j} + 2t\cos t\mathbf{k}$ near $t = \pi/2$.

   (a) Find parameric equations for the line tangent to the curve.

   (b) Find an equation for the plane containing the velocity vector and the acceleration vector.

   (c) Find the center of the *osculating circle* tangent to the curve.

6. † **I'm lying to you, don't you think?**

   In the land of Smullyan, every inhabitant was either right-handed or left-handed; none of the inhabitants were ambidextrous. Moreover, what a right-handed person wrote with his right hand was true, and whatever they wrote with their left hand was false. The left-handed people were the opposite: Whatever they wrote with their left hand was true and whatever they wrote with their right hand was false. In other words, whatever a person wrote with their stronger hand was true and whatever they wrote with their weaker hand was false.

   (a) What statement could be written only by a left-handed person, and using either hand?

   (b) What statement could be written only by a left-handed person using his right hand?

   (c) What statement could be written only by a left-handed person using his left hand?

† Adapted from Raymond Smullyan's *The Riddle of Scheherazade and Other Amazing Puzzles*, Harcourt Brace & Company, 1997, pp. 94-95.