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Math 234 Worksheet 2

Work on problems that you most need practice on. If you are not confident on directional derivatives, tangent planes, or implicit differentiation, be sure to do problems 6 and 7. An asterisk (*) marks more challenging problems.

1. **Quadric surfaces.**
Sketch the following three-dimensional surfaces:
   (a) $\frac{x^2}{1^2}+\frac{y^2}{2^2}+\frac{z^2}{3^2}=1$
   (b) $x^2+y^2=\frac{z^2}{c^2}$
   (c) $z=x^2-y^2$
   (d) $4x^2-y^2+4z^2=4$

   For the last graph, explain how it differs from the graph of $9x^2-y^2+9z^2=9$.

2. **Chain rule for multiple variables.**
Let $f(x,y)=e^{2x}\sin(xy)$ and assume that x and y are functions of s and t as given by $x(s,t)=2s-3t$ and $y(s,t)=s^2-t^3$. Find $\partial f/\partial s$ and $\partial f/\partial t$, expressing them in terms of s and t only.

3. **Magnitude of gradient in polar coordinates.**
Let $z=f(x,y)$, where $x=r\cos\theta$ and $y=r\sin\theta$. Show that

$$\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\left(\frac{\partial z}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$$

   by using the chain rule to expand the right hand side.

4. **Critical points of multivariable functions.**
Back in calculus of one variable, a point x_0 was a critical point for f if $f'(x_0)=0$. Assuming that x_0 was a critical point, if $f''(x_0)>0$ then x_0 was a local minimum and if $f''(x_0)<0$ then x_0 was a local maximum.

   Are there similar rules for functions $z=f(x,y)$? What should be the criteria for a point x_0 to be a critical point? A local minimum or local maximum?

5. **Laplace's equation.**
A function of two variables that satisfies Laplace's Equation

$$f_{xx}+f_{yy}=0$$

   is said to be harmonic.
   (a) Write the equation $f_{xx}+f_{yy}=0$ using ∂ notation.
   (b) Verify that $f(x,y)=x^3y-xy^3$ and $f(x,y)=\ln(4x^2+4y^2)$ are harmonic.
   (c) * Do you think it is possible for a harmonic function to have a local maximum? Why or why not?

6. **Review of directional derivatives.**
Let $f(x,y)=x^2+y^2$ and let P be the point $(-3,4)$.
   (a) Find the linear approximation of f at P.
   (b) Find the directional derivative of f at the point P in the direction toward the origin.
   (c) Find a unit vector in the direction in which f increases most rapidly at P. What is the rate of change of f in this direction?
   (d) Find a unit vector in the direction in which f decreases most rapidly at P. What is the rate of change of f in this direction?
   (e) Find a direction in which the directional derivative is zero. How does this direction relate to level curves of f?

7. **Tangent plane and implicit derivatives.**
Let $F(x,y,z)=x^2-2y^3-3z^4$.
   (a) Find the equation of the tangent plane to the surface $F(x,y,z)=0$ at the point $P=(4,2,0)$.
   (b) The surface $F=0$ implicitly defines a function $z(x,y)$ near P. Find $\frac{\partial z}{\partial x}\big|_P$ and $\frac{\partial z}{\partial y}\big|_P$. (Shortcut: they are the same for the surface and for the tangent plane.)
   (c) * Find a point on the surface $F(x,y,z)=0$ where the tangent plane is parallel to the plane $x+3y-6z=18$. What is the equation of the tangent plane at that point? Is your point unique? If so, why? If not, what other points work?

8. **Tangent plane to ellipsoid.**
Show that the equation of the tangent plane to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

   at the point (x_0,y_0,z_0) can be written in the form

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}=1.$$