1 $y' = 3 + t - y, \quad y(0) = 1$

(d) exact solution : Method of Integrating factors

$$y' + y = 3 + t$$

$$\mu(t) = e^{\int_0^t 1\,ds} = e^t$$

$$\frac{d}{dt}\{e^t y\} = e^t(3 + t)$$

$$e^t y - e^0 1 = \int_0^t 3e^s + se^s\,ds = 3e^s + se^s\Big]_0^t - \int_0^t e^s ds = 2e^s + se^s\Big]_0^t$$

$$= 2e^t + te^t - 1$$

$$\boxed{y = 2 + t - e^{-t}}$$

Numerical Approx. $y_{n+1} = y_n + f(t_n, y_n)\,h$, $f(t_n, y_n) = 3 + t_n - y_n$

(a) $h = 0.1$

n	t_n	y_n
0	0	1
1	0.1	$1 + (3+0-1)\cdot 0.1 = 1.2$
2	0.2	$1.2 + (3+0.1-1.2)\cdot 0.1 = 1.39$
3	0.3	1.5710
4	0.4	1.7439

(b) $h = 0.05$

n	t_n	y_n
0	0	1
1	0.05	1.1
2	0.1	1.1975
3	0.15	1.2926
4	0.2	1.3855
5	0.25	1.4762
6	0.3	1.5649
7	0.35	1.6517
8	0.4	1.7366

(c) $h = 0.025$ similarly to (b)

where $t = 0.1, 0.2, 0.3, 0.4$

corresponds to $n = 4, 8, 12, 16$

t	$y_E(h=0.1)$	$y_E(h=0.05)$	$y_E(h=0.025)$	y exact $(=2+t-e^{-t})$
0	1	1	1	1
0.1	1.2	1.1975	1.1963	1.1952
0.2	1.39	1.3855	1.3833	1.3813
0.3	1.571	1.5649	1.5620	1.5592
0.4	1.7439	1.7366	1.7331	1.7297
error for $t=0.4$	0.0142	0.0069	0.0034	

The error is reduced by a factor of about 2 as h is reduced by a factor of 2

2 $y' = 2y - 1, \quad y(0) = 1$

(d) $1 - \frac{1}{2y-1} \frac{dy}{dt} = 0$

$\frac{d}{dt}\left\{ t - \frac{1}{2} \ln|2y-1| \right\} = 0$

$\left\{ t - \frac{1}{2} \ln|2y-1| \right\} - \left\{ 0 - \frac{1}{2} \ln|\cancel{2}-1| \right\} = 0$

$t = \ln\left(|2y-1|^{\frac{1}{2}}\right)$

$e^t = |2y-1|^{\frac{1}{2}}$, $\quad e^{2t} = 2y-1$ ($e^{2t} = -(2y-1)$ doesn't satisfy initial conditio

$y = \frac{1}{2}e^{2t} + \frac{1}{2}$. (exact solution)

Numerical Approximation.

(a). $h = 0.1$

n	t_n	y_n
0	0	1
1	0.1	1.1
2	0.2	1.22
3	0.3	1.364
4	0.4	1.5368

(b) $h = 0.05$

n	t_n	y_n
0	0	1
1	0.05	1.05
$\boxed{2}$	0.1	1.105
3	0.15	1.1655
$\boxed{4}$	0.2	1.2321
5	0.25	1.3053
$\boxed{6}$	0.3	1.3858
7	0.35	1.4744
$\boxed{8}$	0.4	1.5718

(c) $h = 0.025$

n	t_n	y_n
0	0	1
1	0.025	1.025
2	0.05	1.0513
3	0.075	1.0788
$\boxed{4}$	0.1	1.1078
5	0.125	1.1381
6	0.15	1.1700
7	0.175	1.2036
$\boxed{8}$	0.2	1.2387
9	0.225	1.2757
10	0.25	1.3144
11	0.275	1.3552
$\boxed{12}$	0.3	1.3979
13	0.325	1.4428
14	0.35	1.4900
15	0.375	1.5395
$\boxed{16}$	0.4	1.5914

t	$y_E(h=0.1)$	$y_E(h=0.05)$	$y_E(h=0.025)$	y_{exact}
0	1	1	1	1
0.1	1.1	1.105	1.1078	1.1107
0.2	1.22	1.2321	1.2387	1.2459
0.3	1.364	1.3858	1.3979	1.4111
0.4	1.5368	1.5718	1.5914	1.6128
error for $t=0.4$	0.076	0.041	0.0214	

Similar conclusion to 1.

3. $y' = 0.5 - t + 2y, \quad y(0) = 1$

(a). exact solution. Method of Integrating factors

$$y' - 2y = 0.5 - t$$

$$\mu(t) = e^{\int_0^t -2\,ds} = e^{-2t}$$

$$\frac{d}{dt}\{e^{-2t} \cdot y\} = e^{-2t}(0.5 - t)$$

$$e^{-2t} y - e^0 1 = \int_0^t e^{-2s} 0.5 - s e^{-2s}\, ds = -0.25 e^{-2s} - \left(-\tfrac{1}{2} s e^{-2s} + \tfrac{1}{2}\int_0^t e^{-2s}\, ds\right)$$

$$= \cancel{-0.25 e^{-2s}} + \tfrac{1}{2} \cdot s e^{-2s} + \cancel{\tfrac{1}{4} e^{-2s}} \Big]_0^t = \tfrac{1}{2} t e^{-2t}$$

$$y = \tfrac{1}{2} t + e^{2t}$$

Numerical Approximation. $y_{n+1} = y_n + f(t_n, y_n) h$

Similar to #1, 2,

t	$y_E(h=0.1)$	$y_E(h=0.05)$	$y_E(h=0.025)$	y_{exact}
0	1	1	1	1
0.1	1.25	1.26	1.2655	1.2714
0.2	1.54	1.5641	1.5775	1.5918
0.3	1.878	1.9216	1.9459	1.9721
0.4	2.2736	2.3436	2.3829	2.4255
Error for $t=0.4$	0.1519	0.0819	0.0426	

Similar conclusion to 1.