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Homework Solutions for §3.5

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In problems 1,3,5 find the general solution.

Problem 1. $y'' - 2y' + y = 0$

Characteristic polynomial:
$r^2 - 2r + 1 = 0$
So $(r-1)^2 = 0$
Repeated root $r = 1$, so general solution is:
$y(t) = c_1 e^t + c_2 t e^t$.

Problem 3. $4y'' - 4y' - 3y = 0$

Characteristic polynomial:
$4r^2 - 4r - 3 = 0$
So $(2r+1)(2r-3) = 0$
So $r \in \{-\frac{1}{2}, \frac{3}{2}\}$. So the general solution is:
$y(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{\frac{3}{2}t}$

Problem 5. $y'' - 2y' + 10y = 0$

Guess $y(t) = e^{rt}$.
Characteristic polynomial:
$r^2 - 2r + 10 = 0$
Discriminant is $(-2)^2 - 4 \cdot 10 = 4 - 40 = -36 < 0$, so there are no real roots.
Can solve by quadratic formula.
Or by seeking complex conjugate factors:
Get $(r + (-1-3i))(r + (-1+3i)) = 0$.
i.e. $r = 1 \pm 3i$.
So a complex solution is:
$e^{(-1+3i)t} = e^{-t}(\cos(3t) + i\sin(3t))$.
Since the ODE has real coefficients, the real and imaginary parts of this solution must also be solutions.
The real part is $y_1(t) = e^{-t}\cos(3t)$.
The imaginary part is $y_2(t) = e^{-t}\sin(3t)$.
These are linearly independent solutions.
So the general solution is:
$y(t) = c_1 e^{-t}\cos(3t) + c_2 e^{-t}\sin(3t)$.

In problems 11 and 13 (a) solve the given initial value problem. (b) Sketch the graph of the solution and (c) describe its behavior for increasing t.

Problem 11.

$9y'' - 12y' + 4y = 0$
$y(0) = 2$
$y'(0) = -1$

(a) Solution.

The characteristic polynomial factors as:
$(3r-2)^2 = 0$.
So $r = \frac{2}{3}$.
Repeated root. So general solution is:
$y(t) = c_1 e^{\frac{2}{3}t} + c_2 t e^{\frac{2}{3}t}$.
Observe that $y'(t) = (\frac{2}{3}c_1 + c_2)e^{\frac{2}{3}t} + \frac{2}{3}c_2 t e^{\frac{2}{3}t}$.
Apply initial conditions. Get system:

$$\begin{aligned} 2 &= c_1 \\ -1 &= \frac{2}{3}c_1 + c_2 \end{aligned}$$

So $c_2 = -\frac{7}{3}$.
So the solution is:
$y(t) = 2e^{\frac{2}{3}t} + (-\frac{7}{3})te^{\frac{2}{3}t}$.

(c,b) Behavior as $t \to \infty$ and Graph.

A trick that can help you quickly draw rough graphs of solutions is to determine what the function looks like near $+\infty$, near $-\infty$, and near the initial condition. Let $y_1(t) = 2e^{\frac{2}{3}t}$, and let $y_2(t) = (-\frac{7}{3})te^{\frac{2}{3}t}$.
So $y(t) = y_1(t) + y_2(t)$. Observe that as t goes to $+\infty$, $y_2(t)$ is much bigger than $y_1(t)$. Other ways of saying this are:

- $\lim_{t\to\infty} \frac{y_2(t)}{y_1(t)} = \pm\infty$
- $\lim_{t\to\infty} \frac{y_1(t)}{y_2(t)} = 0$
- $y_2(t)$ dominates $y_1(t)$ as $t \to \infty$.

This means that $y(t)$ looks like $y_2(t)$ as t goes to $+\infty$. Other ways of saying this are:

- $\lim_{t\to\infty} \frac{y(t)}{y_2(t)} = 1$
- $y(t)$ is <u>asymptotic</u> to $y_2(t)$ as $t \to \infty$.
- $y(t) \sim y_2(t)$ as $t \to \infty$.

What happens as t goes to $-\infty$? Certainly y_1, y_2, and y all go to 0. But we can say more precisely how they go to zero. Notice that again $y_2(t)$ is much bigger than $y_1(t)$ as t goes to $-\infty$. So $y(t) \sim y_2(t)$ as $t \to -\infty$.

So graphing $y_2(t)$ can help you graph $y(t)$.

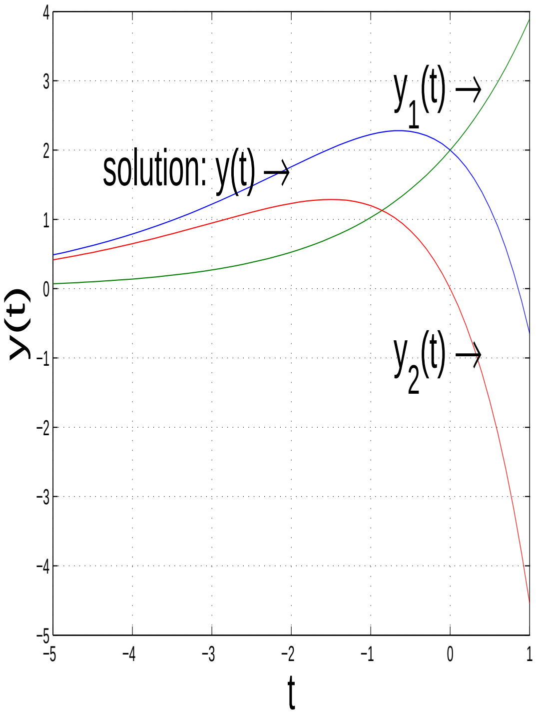

Figure 1: Graph for Problem 11

Problem 13.

$9y'' + 6y' + 82y = 0$
$y(0) = -1$
$y'(0) = -2$

(a) Solution.
The roots of the characteristic polynomial are:
$r = -\frac{1}{3} \pm 3i$. So the general solution is:
$y(t) = c_1 e^{-\frac{1}{3}t}\cos(3t) + c_2 e^{-\frac{1}{3}t}\sin(3t)$.
Note: $y'(t) = c_1(-\frac{1}{3}e^{-\frac{1}{3}t}\cos(3t) - 3e^{-\frac{1}{3}t}\sin(3t)) + c_2(-\frac{1}{3}e^{-\frac{1}{3}t}\sin(3t) + 3e^{-\frac{1}{3}t}\cos(3t))$

The initial conditions give:

$$\begin{aligned} -1 &= c_1 \\ -2 &= c_1(-\frac{1}{3}) + c_2 3 \end{aligned}$$

So $c_2 = \frac{5}{9}$. So the solution is:
$y(t) = e^{-\frac{1}{3}t}(-1\cos(3t)\frac{5}{9}\sin(3t))$.

(c,b) Behavior as $t \to \infty$ and Graph.

To sketch the graph, we rewrite $-1\cos(3t) + \frac{5}{9}\sin(3t)$ in the form $A\cos(3t - \delta)$.
Recall that $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$.
So $A\cos(3t - \delta) = (A\cos\delta)\cos 3t + (A\sin\delta)\sin 3t$.
So we can set:

$$\begin{aligned} A\cos\delta &= -1 \\ A\sin\delta &= \frac{5}{9} \end{aligned}$$

This implies that $A^2 = (-1)^2 + (\frac{5}{9})^2 = \frac{125}{81}$.
So $A = \frac{5\sqrt(5)}{9} \approx 1.242$.
So $y(t) = \frac{5\sqrt(5)}{9}e^{-\frac{1}{3}t}\cos(3t - \delta)$.
We could continue and find δ (by finding $\tan\delta$), but at this point we know that $y(t)$ is an exponentially decaying sinusoid that satisfies the initial conditions and which oscillates within the envelope $\pm e^{-\frac{1}{3}t}$.
This is enough information to sketch a graph (assuming that we don't care about exactly where the zeros are).

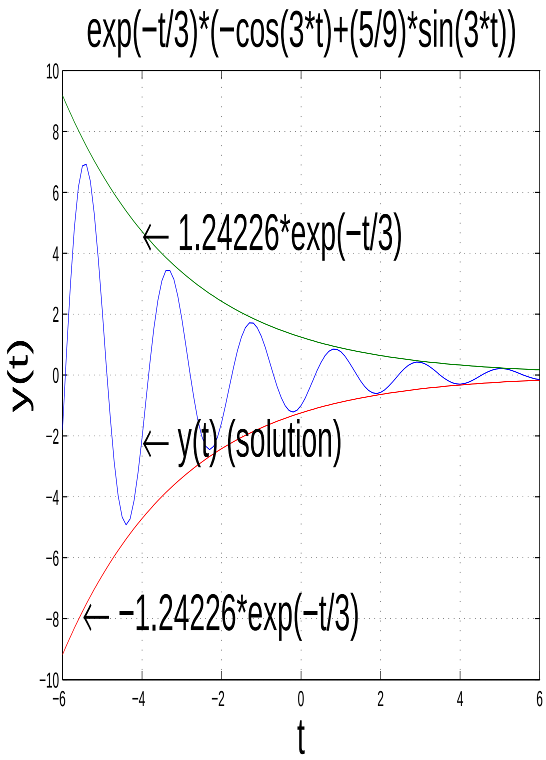

Figure 2: Graph for Problem 13

In problems 25 and 27 use the method of reduction of order to find a second solution of the given differential equation.

Problem 25.

$t^2y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$

Check that this is indeed a solution:
$y_1'(t) = (-1)t^{-2}$ and $y_1''(t) = (2)t^{-3}$.
Substituting shows that y_1 is indeed a solution.

Now divide the differential equation by t^2.
$y'' + 3t^{-1}y' + t^{-2}y = 0$.
Compare with the standard form:
$y'' + p(t)y' + q(t)y = 0$.
So $p(t) = 3t^{-1}$.

According to the method of reduction of order, $v(t)y_1(t)$ is an (independent) solution iff $v(t)$ satisfies:
$y_1v'' + (2y_1' + py_1)v' = 0$.
Separating gives: $-\frac{v''}{v'} = \frac{2y_1'+py_1}{y_1} = \frac{2y_1'}{y_1} + p$.
Integrating gives: $-\ln v' = 2\ln y_1 + \int p + C$.
So $v' = c_1 y_1^{-2} e^{-\int p}$.
In our case, $\int p(t)\ dt = \int 3t^{-1}\ dt = 3\ln t$.
So $v' = c_1(t^{-1})^{-2}e^{-3\ln t} = c_1 t^2 t^{-3} = c_1 t^{-1}$.
So $v(t) = c_1 \ln t + c_2$.
So $v(t)y_1(t) = c_1 t^{-1}\ln t + c_2 t^{-1}$.
This is a linear combination of two linearly independent solutions, so it is the general solution.

Problem 27.

$xy'' - y' + 4x^3y = 0, \quad x > 0; \quad y_1(x) = \sin x^2$

This time let's suppose that all we can remember to do is to make the following guess:
$y(x) = v(x)y_1(x)$.
Substituting into the differential equation and simplifying should give you:
$0 = v'(-\sin x^2 + 4x^2\cos x^2) + v''x\sin x^2$.
Separating gives: $\frac{v''}{v'} = \frac{\sin x^2 - 4x^2\cos x^2}{x\sin x^2} = \frac{1}{x} - \frac{4x\cos x^2}{\sin x^2}$.
Integrating: $\ln v' = \ln x - 2\ln(\sin x^2) + C$
Exponentiating: $v' = c_3 x(sinx^2)^(-2) = c_1 x(\csc x^2)^2$
So $v = c_3 \int x(\csc x^2)^2\ dx$.
Let $u = x^2$. So $du = 2x\ dx$.
So $v = \frac{c_3}{2}\int(\csc u)^2\ du$.
Recall that $\int \csc^2 u\ du = \cot u$.
So $v = c_1\cot(x^2) + c_2$.
Recall that $y = vy_1$.
So $y(x) = c_1\cos(x^2) + c_2\sin(x^2)$.
This is the general solution.