§7.9 p439 # 2, 3, 5

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In each problem find the general solution of the given system of equations.

(2) $\vec{x}' = \underbrace{\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}}_{\text{Call } A} \vec{x} + \underbrace{\begin{pmatrix} e^t \\ \sqrt{3}e^{-t} \end{pmatrix}}_{\text{Call } g(t)}$

A general method to solve the forced linear system

$$\vec{x}' = A\vec{x} + g(t)$$

is to solve the homogeneous problem and then add a particular solution.

To solve the homogeneous problem, one seeks solutions of the form

$$x = \xi e^{rt}.$$

This leads to the eigenvalue-eigenvector problem

$$A\xi = r\xi, \quad \text{i.e.}$$

$$(A - Ir)\xi = 0.$$

Another method is to diagonalize the matrix A.

This also leads to the problem of finding a full set of linearly independent eigenvectors.

Since A is real and symmetric, A is Hermitian, so A has real eigenvalues and a full set of eigenvectors.

Eigenvalues of A

$\det(A - Ir)$

$= \det\begin{pmatrix} 1-r & \sqrt{3} \\ \sqrt{3} & -1-r \end{pmatrix}$

$= r^2 - 4$

$= (r-2)(r+2)$

Eigenvectors

$r_1 = -2: \quad \begin{bmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \overbrace{\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}}^{\vec{\xi}^{(1)}} = \vec{0}$

$r_2 = 2: \quad \begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & -3 \end{bmatrix} \underbrace{\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}}_{\vec{\xi}^{(2)}} = \vec{0}$

Since A is Hermitian, $\vec{\xi}^{(1)}$ and $\vec{\xi}^{(2)}$ are orthogonal.

Indeed: $\vec{\xi}^{(1)} \cdot \vec{\xi}^{(2)} = 0$ ✓.

Form a matrix of eigenvectors:

Let $T = \left[\vec{\xi}^{(1)}\ \vec{\xi}^{(2)}\right] = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$

Let $D = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

Verify that $AT = TD$:

$$\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$LHS = \begin{bmatrix} -2 & 2\sqrt{3} \\ 2\sqrt{3} & 2 \end{bmatrix} = RHS \ ✓$$

Transform the system.

Let $y = T^{-1}x$, i.e. $x = Ty$

Multiply both sides of the ODE by T^{-1}:

$$T^{-1}\vec{x}' = T^{-1}A\vec{x} + T^{-1}\vec{g}$$

So $\vec{y}' = \underbrace{T^{-1}AT}_{D}\vec{y} + T^{-1}\vec{g}$

(2 cont.)

Since the columns of T are orthogonal, we can easily determine its inverse by multiplying by its transpose:

$$T^*T = \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$$\text{So } T^{-1} = \frac{1}{4} T^*$$

$$= \frac{1}{4}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

(You can also normalize the columns of T to get an orthonormal matrix

$$\tilde{T} = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

whose inverse is

$$\tilde{T}^{-1} = \tilde{T}^* = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$)

So our transformed system is:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \left(\frac{1}{4}\right)\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}\begin{pmatrix} e^t \\ \sqrt{3}e^{-t} \end{pmatrix}$$

i.e.

$$\begin{cases} y_1' = -2y_1 + \left(\frac{1}{4}\right)\left[e^t - 3e^{-t}\right] \\ y_2' = 2y_2 + \left(\frac{1}{4}\right)\left[\sqrt{3}e^t + \sqrt{3}e^{-t}\right] \end{cases}$$

This is a decoupled pair of 1st-order linear ODEs, which we can solve by undetermined coefficients (or many other methods).

Homogeneous solutions:

$$y_1^{(h)}(t) = c_1 e^{-2t}$$

$$y_2^{(h)}(t) = c_2 e^{2t}$$

Particular solutions:

$$\text{Seek } Y_1(t) = Ae^t + Be^{-t}$$

$$Y_1'(t) = Ae^t - Be^{-t}$$

$$\text{So } Y_1' + 2Y_1 = 3Ae^t + Be^{-t}$$

$$\text{Need } = \frac{1}{4}e^t - \frac{3}{4}e^{-t}$$

$$\text{So } A = \frac{1}{12},\quad B = -\frac{3}{4}$$

$$\text{So } \boxed{Y_1(t) = \frac{1}{12}e^t - \frac{3}{4}e^{-t}}$$

$$\text{Seek } Y_2(t) = Ae^t + Be^{-t}$$

$$\text{So } Y_2'(t) = Ae^t - Be^{-t}$$

$$\text{So } Y_2' - 2Y_2 = -Ae^t - 3Be^{-t}$$

$$\text{Need } = \frac{\sqrt{3}}{4}e^t + \frac{\sqrt{3}}{4}e^{-t}$$

$$\text{So } A = -\frac{\sqrt{3}}{4},\quad B = \frac{-\sqrt{3}}{12}$$

$$\text{So } Y_2(t) = \frac{-\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t}$$

General solution:

$$y_1(t) = c_1 e^{-2t} + \left(\frac{1}{12}e^t - \frac{3}{4}e^{-t}\right)$$

$$y_2(t) = c_2 e^{2t} + \left(-\frac{\sqrt{3}}{4}e^t - \frac{\sqrt{3}}{12}e^{-t}\right)$$

Transform general solution back to $\vec{x}(t)$:

$$\vec{x}(t) = T\vec{y}(t) = \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}\left[e^{2t}\begin{pmatrix} 0 \\ c_2 \end{pmatrix} + e^{-2t}\begin{pmatrix} c_1 \\ 0 \end{pmatrix} + e^t\begin{pmatrix} \frac{1}{12} \\ -\frac{\sqrt{3}}{4} \end{pmatrix} + e^{-t}\begin{pmatrix} -\frac{3}{4} \\ -\frac{\sqrt{3}}{12} \end{pmatrix}\right]$$

$$= c_2 e^{2t}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} + c_1 e^{-2t}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} + e^t\begin{pmatrix} -2/3 \\ -\sqrt{3}/3 \end{pmatrix} + e^{-t}\begin{pmatrix} -1 \\ \sqrt{3}\,2/3 \end{pmatrix}$$

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③ $\vec{x}' = \underbrace{\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}}_{\text{Call } A} \vec{x} + \underbrace{\begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}}_{\text{Call } \vec{g}(t)}$

Eigenvalues

$$\det(A - Ir) = r^2 - (\operatorname{tr} A) r + \det A = r^2 + 1 = (r-i)(r+i)$$

Eigenvectors

$$r_1 = i: \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \overbrace{\begin{pmatrix} 2+i \\ 1 \end{pmatrix}}^{\xi^{(1)}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$r_2 = -i: \begin{bmatrix} 2+i & -5 \\ 1 & -2+i \end{bmatrix} \begin{pmatrix} 2-i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Fundamental set of homogeneous solutions

$$\vec{x}^{(1)}(t) = \xi^{(1)} e^{r_1 t} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it}$$

$$= \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right](\cos t + i \sin t)$$

$$= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t + i\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin t\right]$$

The real and imaginary parts are both real solutions to the homogeneous equation:

Let $\vec{u}(t) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cos t - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin t$

Let $\vec{v}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \sin t$

The general solution is a linear combination:

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cos t + \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \sin t$$

Particular solution (using undetermined coefficients)

Write $\vec{g}(t) = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t$

Since $r = \pm i$ is an eigenvalue of the coefficient matrix, we guess a solution of the form

$$X(t) = \vec{a} \cos t + \vec{b} \sin t + t(\vec{c} \cos t + \vec{d} \sin t)$$

$$X'(t) = -\vec{a} \sin t + \vec{b} \cos t + \vec{d} \sin t + \vec{c} \cos t + t(-\vec{c} \sin t + \vec{d} \cos t)$$

$$AX + g = A\vec{a} \cos t + A\vec{b} \sin t + t(A\vec{c} \cos t + A\vec{d} \sin t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sin t$$

Matching up coefficients of $\cos t$, $\sin t$, $t\cos t$, and $t \sin t$ gives:

$$\cos t: \quad \vec{b} + \vec{c} = A\vec{a} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\sin t: \quad \vec{d} - \vec{a} = A\vec{b} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left.\begin{aligned} t\cos t: &\quad \vec{d} = A\vec{c} \\ t\sin t: &\quad -\vec{c} = A\vec{d} \end{aligned}\right\}$$

This is a linear system of 8 equations in 8 unknowns.

The last two equations tell us that $\vec{d} = -A^2\vec{d}$, and $\vec{c} = -A^2\vec{c}$

Since $\pm i$ is an eigenvalue of A, -1 is indeed an eigenvector of $\vec{d}$,

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$$A^2 = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

So every vector $\vec{d}$ is an eigenvector of A^2,
So a free choice of $\vec{d}$ determines $\vec{c}$, and vice versa.

We can similarly eliminate $\vec{a}$ or $\vec{b}$ from the first two equations:

$$\vec{b} = A\vec{a} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \vec{c}$$

So:

$$\vec{a} = \vec{d} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - A\left(A\vec{a} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \vec{c}\right)$$

$$= \vec{d} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} - A^2\vec{a} - A\begin{pmatrix} -1 \\ 0 \end{pmatrix} + \underbrace{A\vec{c}}_{\vec{d}}$$

$$\vec{a} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} - A^2\vec{a} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 2\vec{d}$$

i.e. $\underbrace{(A^2 + I)}_{0}\vec{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2\vec{d}$

So $0 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + 2\vec{d}$

So $\boxed{\vec{d} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}}$

Try eliminating $\vec{a}$:

$$\vec{a} = \vec{d} - Ab - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So

$$\vec{b} + \vec{c} = A\left[\vec{d} - A\vec{b} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

So $\cancel{\vec{b}} + \vec{c} = \underbrace{A\vec{d}}_{-\vec{c}} + \underbrace{-A^2\vec{b}}_{\vec{b}} + \underbrace{-A\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\begin{pmatrix} 5 \\ 2 \end{pmatrix}} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

So $2\vec{c} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

So $\boxed{\vec{c} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

Check:

$$A\vec{c} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \vec{d} \checkmark$$

Having found $\vec{d}$ and $\vec{c}$, the first two equations (for $\cos t$ & $\sin t$) simplify to:

$\cos t$: $\vec{b} = A\vec{a} + \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

$\sin t$: $-\vec{a} = A\vec{b} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

We substitute for $\vec{b}$:

$$-\vec{a} = A\left[A\vec{a} + \begin{pmatrix} -3 \\ -1 \end{pmatrix}\right] + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$-\vec{a} = -\vec{a} + \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$0 = 0.$$

So $\vec{a}$ (or $\vec{b}$) may be freely chosen, and $\vec{b}$ (or $\vec{a}$) is then consequently determined.
Let's take $\vec{a}$ as the arbitrary choice,
Write $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

So $\vec{b} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

$$\begin{cases} b_1 = 2a_1 - 5a_2 - 3 \\ b_2 = a_1 - 2a_2 - 1 \end{cases}$$

So our generic particular solution is:

$$X(t) = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\cos t + \begin{pmatrix} 2a_1 - 5a_2 - 3 \\ a_1 - 2a_2 - 1 \end{pmatrix}\sin t$$

$$+ t\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos t + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\sin t\right]$$

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To make our solution look more like the solution in the back of the book, take $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ as the arbitrary choice.

So $\vec{a} = -A\vec{b} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \begin{bmatrix} -2 & 5 \\ -1 & 2 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{cases} a_1 = -2b_1 + 5b_2 - 1 \\ a_2 = -b_1 + 2b_2 - 1 \end{cases}$$

So $X(t) = \begin{pmatrix} -2b_1 + 5b_2 - 1 \\ -b_1 + 2b_2 - 1 \end{pmatrix} \cos t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sin t$

$$+ \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\cos t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t$$

$$\boxed{\begin{aligned} X(t) = {} & b_1 \begin{pmatrix} -2\cos t + \sin t \\ -\cos t \end{pmatrix} + b_2 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} \\ & - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos t \\ & + \begin{pmatrix} 2 \\ 1 \end{pmatrix} t\cos t + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin t \end{aligned}}$$

As a check one ought to be able to show that setting b_1 and b_2 equal to 0 makes this a particular solution, and that

$$\underbrace{-\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}}_{-A} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \cos t + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \sin t$$

is the homogeneous solution, as a simple calculation will verify.

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(5) $\vec{x}' = \underbrace{\begin{pmatrix} 4 & -2 \\ 8 & -4 \end{pmatrix}}_{\text{Call } A} \vec{x} + \underbrace{\begin{pmatrix} t^{-3} \\ -t^{-2} \end{pmatrix}}_{\text{Call } \vec{g}(t)}, \quad t > 0.$

The components of $\vec{g}(t)$ do not fit the pattern of a product of an exponential, a polynomial, and a trigonometric function, so we do not use undetermined coefficients.

Since the problem has constant coefficients, we could apply the Laplace transform.

Let's try the method of variation of parameters.

First obtain a fundamental matrix by obtaining a fundamental set of solutions to the homogeneous equation.

Eigenvalues

$$\det(A - Ir) = r^2 - (\operatorname{tr} A) r + \det A = r^2$$

Eigenvectors

$$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Observe that 0 is an eigenvalue with algebraic multiplicity 2 (it is twice a root of the characteristic polynomial), but geometric multiplicity 1 (it does not have more than 1 linearly independent eigenvector).

So we need to consider generalized eigenvectors.

In accordance with the methods of section 7.8 we seek solutions to the homogeneous equation $\vec{x}' = A\vec{x}$ of the form

$$\vec{x}^{(1)}(t) = \vec{\xi} e^{0t} = \vec{\xi}, \text{ and}$$

$$\vec{x}^{(2)}(t) = \vec{\xi} t e^{0t} + \eta e^{0t} = \vec{\xi} t + \eta$$

The first guess reduces to:

$A\vec{\xi} = 0$, which we have solved.

The second guess gives:

$$\vec{\xi} = A(\vec{\xi} t + \vec{\eta})$$

So $A\vec{\xi} = 0$

and $A\vec{\eta} = \vec{\xi}$.

So $\vec{\eta}$ needs to be a generalized eigenvector for 0.

Need $\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \vec{\xi}$

So take $\vec{\eta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So $x^{(1)}(t) = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

and $x^{(2)}(t) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

constitute a fundamental set.

So a fundamental matrix is:

$$\Psi(t) = \left[x^{(1)}(t) \;\; x^{(2)}(t) \right] = \begin{bmatrix} 2 & 2t+1 \\ 4 & 4t+1 \end{bmatrix}$$

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Check:

$$\psi'(t) = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$$

$$A\psi(t) = \begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}\begin{bmatrix} 2 & 2t+1 \\ 4 & 4t+1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} \checkmark$$

Particular solution to $x' = Ax + g$.

Variation of parameters tells us to seek a solution of the form

$$\vec{x}(t) = \psi(t)\,\vec{u}(t),$$

where $\vec{u}(t)$ is an unknown function.

$$\text{So } \vec{x}' = \psi'\vec{u} + \psi\vec{u}'$$

$$\text{But } A\vec{x} + g = \underbrace{A\psi}_{=\psi'} u + \vec{g}$$

$$\text{So need } \psi\vec{u}' = \vec{g}$$

$$\text{i.e. } \vec{u}' = \psi^{-1}\vec{g},$$

$$\text{i.e. } \vec{u} = \int(\psi^{-1}\vec{g}).$$

Recall the formula for the inverse of a 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

$$\text{So } \psi^{-1} = \begin{bmatrix} 2 & 2t+1 \\ 4 & 4t+1 \end{bmatrix}^{-1} = \frac{-1}{2}\begin{bmatrix} 4t+1 & -2t-1 \\ -4 & 2 \end{bmatrix}$$

$$\text{So } \vec{u}' = \psi^{-1}\vec{g} = \frac{1}{2}\begin{bmatrix} 4t+1 & -2t-1 \\ -4 & 2 \end{bmatrix}\begin{pmatrix} -t^{-3} \\ t^{-2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -t^{-3} - 5t^{-2} - 2t^{-1} \\ +4t^{-3} + 2t^{-2} \end{pmatrix}$$

$$\text{So } \vec{u} = \frac{1}{2}\begin{pmatrix} \frac{1}{2}t^{-2} + 5t^{-1} - 2\ln t + c_1 \\ -2t^{-2} - 2t^{-1} + c_2 \end{pmatrix}$$

$$= t^{-2}\begin{pmatrix} \frac{1}{4} \\ -1 \end{pmatrix} + t^{-1}\begin{pmatrix} \frac{5}{2} \\ -1 \end{pmatrix} + \ln t\begin{pmatrix} -1 \\ 0 \end{pmatrix} + c_1\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}$$

$$\text{So } \vec{x}(t) = \psi(t)\,u(t) = \begin{bmatrix} 2 & 2t+1 \\ 4 & 4t+1 \end{bmatrix}\vec{u}(t)$$

$$= \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}\vec{u}(t) + t\begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}\vec{u}(t)$$

$$= t^{-2}\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + t^{-1}\begin{pmatrix} 4 \\ 9 \end{pmatrix} + \ln t\begin{pmatrix} -2 \\ -4 \end{pmatrix} + c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$+ t\left[t^{-2}\begin{pmatrix} -2 \\ -4 \end{pmatrix} + t^{-1}\begin{pmatrix} -2 \\ -4 \end{pmatrix} + \ln t\begin{pmatrix} 0 \\ 0 \end{pmatrix} + c_1\begin{pmatrix} 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]$$

So:

$$\boxed{\begin{aligned} x(t) &= t^{-2}\begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix} + t^{-1}\begin{pmatrix} 2 \\ 5 \end{pmatrix} + \ln t\begin{pmatrix} -2 \\ -4 \end{pmatrix} \\ &\quad + \underbrace{\begin{pmatrix} -2 \\ -4 \end{pmatrix} + c_1\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{\tilde{c}_1\begin{pmatrix} 1 \\ 2 \end{pmatrix},\ \text{where } \tilde{c}_1 = c_1 - 2} + c_2\begin{pmatrix} t+\frac{1}{2} \\ 2t+\frac{1}{2} \end{pmatrix} \\ &\text{Back of book has} \\ &\hat{c}_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} t \\ 2t-\frac{1}{2} \end{pmatrix} \\ &\text{To see that this is the same, let } \hat{c}_1 = \tilde{c}_1 + \tfrac{1}{2}c_2 \end{aligned}}$$