Decomposition into perpendicular and parallel components

Thm Let V be a vector space with an inner product,
Let $w_1, \dots, w_n \in V$ span a subspace $W \subseteq V$, and let $v \in V$.
I claim that v can be written uniquely as a sum of two vectors, $v_{/\!/} \in W$, and $v_\perp$ orthogonal to W.
i.e. $\exists!\, v_{/\!/}, v_\perp \in V$ s.t. $v = v_{/\!/} + v_\perp$, $v_{/\!/} \in W$, and $v_\perp \cdot w = 0 \;\; \forall w \in W$.

Lemma Add to the previous theorem the hypothesis that $w_i \cdot w_j = 0$ if $i \neq j$. (Gram-Schmidt idea).

Uniqueness:

Suppose: $v = v_{/\!/} + v_\perp$
where $v_{/\!/} \in W$ and $v_\perp \cdot w = 0 \;\; \forall w \in W$.

Write $\boxed{v_{/\!/} = \lambda_1 w_1 + \dots + \lambda_n w_n}$

$v \cdot w_i = \lambda_i (w_i \cdot w_i)$

So $\boxed{\lambda_i = \dfrac{v \cdot w_i}{w_i \cdot w_i}}$

$v_\perp = v - v_{/\!/}$.

So assuming this expansion exists, we can calculate it.

Existence

Let $v_{/\!/} = \sum_i \left(\dfrac{v \cdot w_i}{w_i \cdot w_i}\right) w_i$

Let $v_\perp = v - v_{/\!/}$

Need:
- $v = v_{/\!/} + v_{/\!/}$ ✓
- $v_{/\!/} \in W$ ✓
- $v_\perp \cdot w = 0 \;\; \forall w \in W$

Let $w \in W$.
Write $w = \mu_1 w_1 + \dots + \mu_n w_n$

$$v_\perp \cdot w = (v - v_{/\!/}) \cdot w$$
$$= \left[v - \left(\sum_i \left(\frac{v \cdot w_i}{w_i \cdot w_i}\right) w_i\right)\right] \cdot \left[\sum_j \mu_j w_j\right]$$
$$= \sum_j \mu_j v \cdot w_j - \sum_j \mu_j w_j \cdot \left(\sum_i \left(\frac{v \cdot w_i}{w_i \cdot w_i}\right) w_i\right)$$
$$= \sum_j \mu_j v \cdot w_j - \sum_j \mu_j v \cdot w_j \left(\frac{w_j \cdot w_j}{w_j \cdot w_j}\right)$$
$$= 0, \text{ as desired.}$$

Pf of thm By the lemma, it suffices to show that an orthogonal basis exists for W.
But the lemma show how to construct such a basis inductively.