Summary: Calculus of Variations

(0): Simplest case.

(a) Let $J(u) = \int_{x_1}^{x_2} f(u, u_x, x)\,dx$

Suppose u minimizes J.
Let v be a test perturbation,
So $v(x_1) = v(x_2) = 0$.

$$0 = \frac{d}{d\varepsilon} J(u+\varepsilon v)\Big|_{\varepsilon=0} = \int_{x_1}^{x_2} v f_u + v_x f_{u_x}\,dx$$

$$= \Big[\overset{0}{\cancel{v} f_{u_x}}\Big]_{x_1}^{x_2} + \int_{x_1}^{x_2} v\left(f_u - \tfrac{d}{dx} f_{u_x}\right)dx$$

Since v is arbitrary,

$$(*)\quad \boxed{f_u - \frac{d}{dx} f_{u_x} = 0} \quad \text{(Euler equation)}$$

(b) Alternate form.

Want to write the Euler equation in the form: $\frac{d}{dx}[\text{something}] = 0$.

Observe:
$$\frac{d}{dx} f = f_x + u_x f_u + u_{xx} f_{u_x}$$

Strategy: multiply the Euler equation by u_x and use parts,

Get: $u_x f_u - u_x \frac{d}{dx} f_{u_x} = 0$

$$u_x f_u - \left[\frac{d}{dx}\left(u_x f_{u_x}\right) - u_{xx} f_{u_x}\right] = 0$$

i.e.
$$\underbrace{\left(u_x f_u + u_{xx} f_{u_x}\right)}_{\frac{d}{dx} f - f_x} - \frac{d}{dx}\left(u_x f_{u_x}\right) = 0$$

So: $\boxed{f_x - \frac{d}{dx}\left[f - u_x f_{u_x}\right] = 0}$

If f is independent of x, this gives:

$$\boxed{f - u_x f_{u_x} = \text{constant}}$$

Note: This seems to generalize (hideously) to higher order derivatives

This does not seem to generalize to more variables.

Problem: $\frac{d}{dx} f = f_x + u_1' f_{u_1} + u_1'' f_{u_1'}$
$\qquad + u_2' f_{u_2} + u_2'' f_{u_2'}$

More independent variables?

(1) More variables ($\Rightarrow$ System of equations)

Let $\boxed{J(\underline{u}) = \int_{x_1}^{x_2} f(x; u_1, \dots, u_n, u_1', \dots, u_n')}$

Suppose $\underline{u}(x)$ minimizes J.
Let v_i be a test perturbation of u_i.
So $v_i(x_1) = v_i(x_2) = 0$.

$$0 = \frac{d}{d\varepsilon} J(\underline{u} + \varepsilon v_i \underline{e}_i)\Big|_{\varepsilon=0} = \int_{x_1}^{x_2} \frac{d}{d\varepsilon} f(x, u_1, \dots, u_i + \varepsilon v_i, \dots, u_n, u_1', \dots, u_i' + \varepsilon v_i', \dots)$$

$$= \int_{x_1}^{x_2} v_i f_{u_i} + v_i' f_{u_i'}\,dx$$

$$= \int_{x_1}^{x_2} v_i\left(f_{u_i} - \frac{d}{dx} f_{u_i'}\right)dx$$

So we get n coupled ODEs:

$$\boxed{f_{u_i} - \frac{d}{dx} f_{u_i'} = 0}$$

These are 2nd order ODEs.
Specifying $u_i(x_1)$ and $u_i(x_2)$ gives the $2n$ many boundary conditions we need.

(2) More independent variables ($\Rightarrow$ PDE)

Let $\boxed{J(u) = \int_\Omega f(\underline{x}; u, \nabla u)}$ [Note: $[\nabla u]_i = \partial_i u = u_{x_i} = u_{,i}$]

Suppose $u(\underline{x})$ minimizes J.
Let v be a test perturbation. So $v|_{\partial\Omega} = 0$.

$$0 = \frac{d}{d\varepsilon} J(u + \varepsilon v)\Big|_{\varepsilon=0} = \int_\Omega \frac{d}{d\varepsilon} f(\underline{x}, u+\varepsilon v, \nabla u + \varepsilon \nabla v)$$

$$= \int_\Omega f_u v + \sum_i f_{\partial_i u}\, \partial_i v$$

$$= \int_\Omega f_u v + \sum_i \left[\partial_i\left(f_{\partial_i u} v\right) - v\, \partial_i f_{\partial_i u}\right]$$

$$= \left(\sum_i \int_{\partial\Omega} n_i \overset{0}{v} f_{\partial_i u}\right) + \int_\Omega v\left(f_u - \sum_i \partial_i f_{\partial_i u}\right)$$

Since v is arbitrary, we get the PDE:

$$\boxed{f_u - \sum_i \partial_{x_i} f_{\partial_{x_i} u}}$$

Boundary conditions: specify $u|_{\partial\Omega}$.

Holding x_j constant $\forall j \neq i$ (i.e. viewing $f_{u_i}(\underline{x})$)

$\Leftrightarrow f_u - \nabla \cdot f_{\nabla u}$

(3) Higher order derivatives ($\Rightarrow$ Higher order DE)

Let $\boxed{J(u) = \int_{x_1}^{x_2} f(x; u, u', \dots, u^{(n)})}$

Let $0 = \frac{d}{d\varepsilon} J(u+\varepsilon v)\Big|_{\varepsilon=0} = \int_{x_1}^{x_2} f_u v + f_{u'} v' + \dots + f_{u^{(n)}} v^{(n)}$

[Require $v^{(i)}(x_1) = 0 = v^{(i)}(x_2)\quad \forall i \in \mathbb{Z}_n$]

So $0 = \int_{x_1}^{x_2} v\left(f_u - \frac{d}{dx} f_{u'} + \dots + (-1)^n \frac{d^n}{dx^n} f_{u^{(n)}}\right)$

So
$$\boxed{\sum_{k=0}^{n} (-1)^k \left(f_{u^{(k)}}\right)^{(k)} = 0}$$

This is an order $2n$ ODE.
BCs: specify $u^{(k)}(x_1)$ and $u^{(k)}(x_2)\quad \forall k \in \mathbb{Z}_n$

If $J(u) = \int_{x_1}^{x_2} f(x; u, u', u'')$, get:

$$\boxed{f_u - \frac{d}{dx} f_{u'} + \frac{d^2}{dx^2} f_{u''} = 0}$$

(4^{th}-order)

Summary of Variational Calculus: arcane forms

(3b)

Alternate form for higher order derivatives:

$$\frac{d}{dx} f = f_x + \sum_{k=0}^{n} f_{u^{(k)}} D^{k+1} u$$

Strategy: multiply the Euler equation by u_x and use parts:

$$0 = \sum_{k=0}^{n} (-1)^k D^k f_{u^{(k)}} Du$$

Idea is to transfer all k derivatives to u and then use the formula above.

Generalizing parts:

$$(Dg)h = D(gh) - g\,Dh$$

$$(D^2 g)h = D(Dg\,h) - Dg\,Dh$$

$$= D(Dg\,h) - D(g\,Dh) + g\,D^2 h$$

$$(D^k g)h = (-1)^k g\,D^k h + \sum_{\ell=1}^{k} D(D^{k-\ell} g\, D^{\ell-1} h)(-1)^{\ell-1}$$

$$\therefore D^k f_{u^{(k)}} Du = (-1)^k f_{u^{(k)}} D^{k+1} u + \sum_{\ell=1}^{k} D(D^{k-\ell} f_{u^{(k)}} D^{\ell} u)(-1)^{\ell-1}$$

So Euler becomes:

$$0 = Df - f_x + \sum_{k=0}^{n} \sum_{\ell=1}^{k} (-1)^{k+\ell-1} D(D^{k-\ell} f_{u^{(k)}} D^{\ell} u)$$

If $f_x = 0$, then factor out D to get:

$$\boxed{C = f + \sum_{k=0}^{n} \sum_{\ell=1}^{k} (-1)^{k+\ell-1} D^{k-\ell} f_{u^{(k)}} D^{\ell} u}$$

For example, if $n = 1$, we get:

$$C = f + (-1)(Du) f_{u^{(k)}} \quad ✓$$

If $n=2$, we get:

$$C = f + \underbrace{(-1)(Du) f_{u^{(1)}}}_{k=1,\ \ell=1} + \underbrace{D f_{u^{(2)}} Du}_{k=2,\ \ell=1} + \underbrace{(-1) f_{u^{(2)}} D^2 u}_{k=2,\ \ell=2}$$

i.e.

$$\boxed{C = f - u' f_{u'} + u' \frac{d}{dx} f_{u''} - u'' f_{u''}}$$

Agrees with Gelfand & Fomin p52#13

(This is a 3rd order differential equation)

(2b)

Alternate form for more independent variables