Summary:

Lagrange Multipliers for variational problems with integral constraints

Problem:

Extremize $J(u) = \int_{t_1}^{t_2} f(t,u,\dot{u})$

subject to $K_i(u) = \int_{t_1}^{t_2} g_i(t,u,\dot{u}) = 0.$

Solution:

Let u be a stationary function of J satisfying the constraint $K_i(u)=0$.

Let v be a test perturbation.

v is an "allowed" perturbation if

$$\frac{d}{d\varepsilon} K_i(u+\varepsilon v)\Big|_{\varepsilon=0} = 0.$$

J is stationary at u for the perturbation v if:

$$\frac{d}{d\varepsilon} J(u+\varepsilon v)\Big|_{\varepsilon=0} = 0.$$

But $\frac{d}{d\varepsilon} J(u+\varepsilon v)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_{t_1}^{t_2} f+\varepsilon(v f_u + \dot{v} f_{\dot{u}}) + O(\varepsilon^2)\Big|_{\varepsilon=0}$

$= \int_{t_1}^{t_2} v\left(f_u - \frac{d}{dt} f_{\dot{u}}\right)$

$= \langle v, \nabla J\rangle$

where $\nabla J \equiv J_u - \frac{d}{dt} J_{\dot{u}}$.

So $J(u+\varepsilon v) = J(u) + \varepsilon\langle \nabla J, v\rangle + O(\varepsilon^2)$.

So to map onto the abstract problem we note that:

∇J = variational derivative

inner product is given by:

$h_1 \cdot h_2 \equiv \int_{t_1}^{t_2} h_1 h_2$.

So it will suffice to "stationize"

$$\boxed{L(u,\lambda) = J(u) + \sum_i \lambda_i K_i(u)}$$

Indeed, this gives:

$$\boxed{\begin{aligned} 0 &= \nabla L = \nabla J + \sum_i \lambda_i \nabla K_i \Big|_u \\ 0 &= K_i\Big|_u \end{aligned}}$$

To recapitulate, the problem becomes:

Stationize

$$L[u,\lambda] = \int_{t_1}^{t_2} f(t,u,\dot{u}) + \sum_i \lambda_i g_i(t,u,\dot{u})\, dt$$

Corresponding abstract problem

Problem:

Extremize $J(X)$

subject to $K_i(X) = 0$

Note: $X \in V$, an "abstract" vector space with an inner product, $J, K: V \to \mathbb{R}$.

Solution:

Let X be a stationary vector of J satisfying the constraint $K_i(X)=0$.

Let ΔX be a test perturbation.

ΔX is an allowed perturbation if

$$\frac{d}{d\varepsilon} K_i(X+\varepsilon\Delta X)\Big|_{\varepsilon=0} = 0$$

Assume that J and K are differentiable.

This means that near any point X, J (or K) may be approximated by an inner product with a vector which we call $\nabla J(X)$. Specifically:

$$J(X+\varepsilon\Delta X) = J(X) + \varepsilon \nabla J(X)\cdot\Delta X + O(\varepsilon^2)$$

So ΔX is an allowed perturbation if

$\nabla K_i(X)\cdot\Delta X = 0$ (i.e. ΔX is perpendicular to $\nabla K(X)$)

If ΔX is an allowed perturbation then J is a stationary point of X, i.e.

$\nabla J(X)\cdot\Delta X = 0.$ (i.e. ΔX is perpendicular to $\nabla J(X)$)

Thus:

If $\nabla K_i(X)\cdot\Delta X = 0 \quad \forall i$

Then $\nabla J(X)\cdot\Delta X = 0$

(i.e. if ΔX is $\perp$ to every $\nabla K_i(X)$, then ΔX is $\perp$ to $\nabla J(X)$)

But this is true iff $\nabla J(X)$ is in the linear span of the constraint gradients $\nabla K_i(X)$.

To see this:

Decompose ∇J and ΔX into components parallel and perpendicular to the subspace spanned by the constraint gradients. (This can be done uniquely).

So write:

$\nabla J = \nabla J_\perp + \nabla J_\parallel$

$\Delta X = \Delta X_\perp + \Delta X_\parallel$

Then

$$\nabla J\cdot\Delta X = \underbrace{\underbrace{\nabla J_\perp}_{?0}\cdot\underbrace{\Delta X_\perp}_{\text{free},\,\neq 0}}_{0} + \underbrace{\underbrace{\nabla J_\parallel}_{\text{free}}\cdot\underbrace{\Delta X_\parallel}_{0}}_{0}$$

So LHS will be $0 \;\; \forall \Delta X_\perp$ precisely when $\nabla J_\perp$ is zero.

So $\nabla J = \nabla J_\parallel$. So can write:

$$\boxed{\nabla J(X) = \sum_i \lambda_i \nabla K_i(X)}$$

Summary:
Lagrange multipliers for variational problems with $\boxed{\text{pointwise constraints}}$

Problem:
Extremize $J(u) = \int_{t_1}^{t_2} f(t,u,\dot{u})$
subject to $g(t,u,\dot{u}) = 0$

Note: For smooth g,
$g(t,u,\dot{u}) = 0$
$\Leftrightarrow \forall$ test function $c(t)$
$\int_{t_1}^{t_2} c(t)\, g(t,u,\dot{u}) = 0$

$$\boxed{\begin{array}{l}\Leftrightarrow K_\tau(u) = 0 \quad \forall \tau \\ \text{where } K_\tau(u) = \int_{t_1}^{t_2} \delta(t-\tau)\, g(t,u,\dot{u}) = 0\end{array}}$$

Idea:
We want to use the result for integral constraints to obtain a result for this problem that approximates the true extremum arbitrarily well. (Note that the approximation will actually be slightly less constrained, so a minimum a little too low.) As long as the approximation extrema aren't too wild, I expect to get the correct answer in the limit.

Solution:
Pick discrete values of τ.
Then it suffices to "stationize"

$$L(u,\tilde{\lambda}) = J(u) + \sum_\tau \tilde{\lambda}_\tau K_\tau(u)$$
$$= \int_{t_1}^{t_2} f(t,u,\dot{u})\,dt + \sum_\tau \tilde{\lambda}_\tau \int_{t_1}^{t_2} \delta(t-\tau)\, g(t,u,\dot{u})\,dt$$
$$= \int_{t_1}^{t_2} f(t,u,\dot{u})\,dt + \sum_\tau \tilde{\lambda}_\tau\, g(\tau,u,\dot{u})$$
$$\approx \int_{t_1}^{t_2} f(t,u,\dot{u})\,dt + \int_{t_1}^{t_2} \lambda(\tau)\, g(\tau,u,\dot{u})\,d\tau$$

[where $\lambda_\tau = \lambda(\tau)\cdot$(grid spacing)]

$$\boxed{L[u,\lambda] = \int_{t_1}^{t_2} f(t,u,\dot{u}) + \lambda(t)\, g(t,u,\dot{u})\, dt}$$

Let E denote the variational derivative (i.e. the Euler operator.)
We want to find $u(t)$ and $\lambda(t)$ to make $L[u,\lambda]$ stationary.
Suppose u and λ constitute a stationary point.

Then holding u or λ constant, $L[u,\lambda]$ is stationary.

Our extremizing problem becomes:
Find $u(t)$, $\lambda(t)$ such that:

① $E[f(t,u,\dot{u}) + \lambda(t)\, g(t,u,\dot{u})] = 0$
(holding $\lambda(t)$ constant and minimizing over $u(t)$.)

② $0 = g(t,u,\dot{u})$ unless $\lambda(t) = 0$
(holding $u(t)$ constant and varying $\lambda(t)$, this is the only way $\lambda(t)$ can be stationary.)

To cover the case $\lambda(t)=0$, I'll be more formal:
Suppose $u(t)$ is given. Write:

$$L[u,\lambda] := \int_{t_1}^{t_2} f[u](t) + \lambda(t)\, g[u](t)\, d$$
$$= \int_{t_1}^{t_2} h(t;\lambda(t))\, dt$$

Require $\frac{\delta L}{\delta \lambda} = 0$

But $\frac{\delta L}{\delta \lambda} = h_\lambda = g[u](t)$

So $g[u](t) = 0$, as desired.