Equivalence of Axiom of Choice with Zorn's Lemma and Hausdorff maximality principle

Def Let X be a set.
The <u>power set</u> of X, denoted $\mathcal{P}(X)$, is the collection of all subsets of X.

Def
$f : \mathcal{P}(X) \to X$ is called a <u>choice function for X</u> if
$(\forall E \in \mathcal{P}(X))\ f(E) \in E$.

Def
Let $\mathcal{S}$ be a collection of sets.
$f : \mathcal{S} \to \bigcup \mathcal{S}$ is called a <u>choice function</u> for $\mathcal{S}$ if
$(\forall S \in \mathcal{S})\ f(S) \in S$.

(C) **Axiom of choice**
(1) Every set X has a choice function.
(2) Every collection of sets $\mathcal{S}$ has a choice function.

These two formulations of the axiom of choice are obviously equivalent.

Def
Let P be a set.
Let $\leq$ be a binary relation on P
Let $\leq$ be a (a mapping of $P \times P$ to $\mathcal{B}$, the set of truth values)
The P is a <u>partially ordered set</u> or <u>poset</u> with respect to $\leq$ provided $\forall a, b, c \in P$
(i) $a \leq a$ (reflexivity)
(ii) $a \leq b$ & $b \leq c \Rightarrow a \leq c$ (transitivity)
(iii) $a \leq b$ & $b \leq a \Rightarrow a = b$ (symmetry)

Def Two elements $a, b \in P$ are <u>comparable</u> if $a \leq b$ or $b \leq a$.

Def A poset P is <u>linearly ordered</u> (or <u>totally ordered</u>) if every two elements are comparable.

Def
Let P be a poset.
Let $S \subset P$ be a subset.
Let $u \in P$. We say u is an <u>upper bound</u> for S if $\forall x \in S\ \ x \leq u$.

Def
$m \in P$ is a <u>maximal element</u> if $\nexists x \in P$ s.t. $x > m$,
(i.e. $m \leq x$ and $x \nleq m$).

(Z) **Zorn's lemma**
A nonempty poset which contains an upper bound for every linearly ordered subset has a maximal element.

(H) **Hausdorff Maximality Theorem**
Every (nonempty) partially ordered set P contains a maximal linearly ordered set.

Proofs of equivalence

$(Z) \Rightarrow (H)$
Let P be a poset, and let $\mathcal{L}$ be the poset of all linearly ordered subsets of P under the ordering of set inclusion.
A linearly ordered subset of $\mathcal{L}$ is called a subchain of $\mathcal{L}$.
The union of a subchain of $\mathcal{L}$ is in $\mathcal{L}$.
The union of a subchain satisfies the hypothesis of (Z).
So $\mathcal{L}$ satisfies the hypothesis of (Z).
So $\mathcal{L}$ has a maximal element L.
Such an L is a maximal linearly ordered set of P.

$(H) \Rightarrow (Z)$
Let P be a poset which contains an upper bound for every linearly ordered subset.
Let M be a maximal linearly ordered set.
Let u be an upper bound for M.
Suppose $u < m$. By transitivity, $M \cup \{m\}$ is a linearly ordered proper superset of M ✗.

(Equivalents of Axiom of Choice)

$(Z) \Rightarrow (C)$

Let X be a set.
Let P be the poset of all functions $f : \mathcal{P}(X) \to \mathcal{P}(X)$ such that $\emptyset \notin \text{range}(f)$ and $f(Y) \subset Y \ \forall Y \in \mathcal{P}(X)$, under the partial ordering $f_1 \geq f_2$ if $f_1(Y) \subset f_2(Y) \ \forall Y \in \mathcal{P}(X)$.

Clearly P is nonempty, since $(Y \mapsto Y) \in P$.

Let f_0 be a maximal element of P.

Claim: $(\forall Y \in \mathcal{P}(X))$, $\# f_0(Y) = 1$, where $\# S$ denotes the cardinality of the set S.

Suppose not. Let $\# f_0(\tilde{Y}) > 1$.
Let $\tilde{y} \in \tilde{Y}$.
Let $\tilde{f}_0 : Y \mapsto \begin{cases} Y & \text{if } Y \neq \tilde{Y} \\ \tilde{Y} \setminus \tilde{y} & \text{if } Y = \tilde{Y} \end{cases}$

Then $\tilde{f}_0 < f_0$ #.

Let $f : \mathcal{P}(X) \to X$,
$Y \mapsto$ the element of $f_0(Y)$.

Then f is a choice function for X.

$(C) \Rightarrow (Z)$ (See Isaacs p154)

Hausdorff maximality thm (Rudin RA)

Def
Let $\mathcal{F}$ be a collection of sets.
Say $\Phi \subset \mathcal{F}$ is a <u>subchain</u> of $\mathcal{F}$ if Φ is ordered by set inclusion.

Def
<u>union</u> of Φ := union of all members of Φ.

Lemma
$\mathcal{F}$ nonempty collection of subsets of a set X such that
$\mathcal{F}$ is closed under unions of subchains.
$g: \mathcal{F} \to \mathcal{F}$ s.t. $\forall A \in \mathcal{F}$
$A \subset g(A)$ and
$\#(g(A) \setminus A) \leq 1$ ← cardinality
(Say $g(A)$ is the "successor" of A.)

Then
$\exists A \in \mathcal{F}$ s.t. $g(A) = A$.

Pf
Fix $A_0 \in \mathcal{F}$.
Say $\mathcal{F}' \subset \mathcal{F}$ is a <u>tower</u> (over A_0) if
(a) $A_0 \in \mathcal{F}'$
(b) $\mathcal{F}'$ is closed under unions of subchains.
(c) $\mathcal{F}'$ is closed under g
(i.e. $A \in \mathcal{F}' \Rightarrow g(A) \in \mathcal{F}'$)

The family of all towers is nonempty.
Indeed, $\{A \in \mathcal{F} : A_0 \subset A\}$ is a tower.
Let $\mathcal{F}_0$ be the intersection of all towers.
$\mathcal{F}_0$ is a tower.
Also, $A_0 \subset A$ if $A \in \mathcal{F}_0$.
Want $\mathcal{F}_0$ is a subchain of $\mathcal{F}$.

Let $\Gamma = \{C \in \mathcal{F}_0 : (\forall A \in \mathcal{F}_0)\ A \subset C \text{ or } C \subset A\}$
Want $\Gamma = \mathcal{F}_0$.

$(\forall C \in \Gamma)$ let $\Phi(C)$
$= \{A \in \mathcal{F}_0 : A \subset C \text{ or } g(C) \subset A\}$

Want that Γ is a tower.
Want that $\Phi(C)$ is a tower.

Properties (a) and (b) are "clearly" satisfied by Γ and by each $\Phi(C)$.

Fix $C \in \Gamma$. Want $g(C) \in \Gamma$.
Let $A \in \Phi(C)$.
Want $g(A) \in \Phi(C)$.
Need $g(A) \subset C$ or $g(C) \subset g(A)$.
Since $A \in \Phi(C)$, have 3 possibilities:
① $A \subset C$ and $A \neq C$,
② $A = C$, or
③ $g(C) \subset A$.

<u>Case $A \subsetneq C$</u>:
Then C is not a proper subset of $g(A)$.
Since $C \in \Gamma$, $C \subset g(A)$ or $g(A) \subset C$.
So $g(A) \subset C$.

<u>Case $A = C$</u>: then $g(A) = g(C)$
<u>Case $g(C) \subset A$</u>: then $g(C) \subset g(A)$.

So we have verified that $g(A) \in \Phi(C)$.
So $\Phi(C)$ satisfies (c), so is a tower.
So $\Phi(C) = \mathcal{F}_0$.
So $(\forall A \in \mathcal{F}_0)(\forall C \in \Gamma)$ $A \subset C$ or $g(C) \subset A$.
So $g(C) \in \Gamma$. So Γ satisfies (c).
So Γ is a tower. But $\Gamma \subset \mathcal{F}_0$,
So $\Gamma = \mathcal{F}_0$. So by the def. of Γ,
$\mathcal{F}_0$ is totally ordered.
So $\mathcal{F}_0$ is a subchain of $\mathcal{F}_0$
So the union A of $\mathcal{F}_0$ is in $\mathcal{F}_0$.
By (c), $g(A) \in \mathcal{F}_0$. So $g(A) \subset A$.
So $g(A) = A$, as desired.

(C) $\Rightarrow$ (H)

Let P be a nonempty poset.
Need P has a maximal totally ordered set.
Let $\mathcal{F}$ be the collection of all totally ordered subsets of P.
$\mathcal{F}$ contains a singleton, so is nonempty.
Call a collection of sets totally ordered by inclusion a chain.
Note that the union of any chain of totally ordered sets is totally ordered.
Let f be a choice function for P.
For $A \in \mathcal{F}$, let A^* be the set of all x in the complement of A such that $A \cup \{x\} \in \mathcal{F}$.

Let $g(A) = \begin{cases} A \cup f(A^*) & \text{when } A^* \neq \emptyset \\ A & \text{when } A^* = \emptyset \end{cases}$

$\mathcal{F}$ and g satisfy the hypotheses of the Hausdorff maximality lemma,
So $\exists A$ s.t. $g(A) = A$,
i.e. $A^* = \emptyset$, so
A is a maximal element of $\mathcal{F}$.

Thm

L vector space

H_1, H_2 Hamel bases for X.

claim $\# H_1 = \# H_2$.

Pf Use transfinite induction.

Let $\mathcal{L}$ be the collection of all linear subspaces of L.

Let $\mathcal{L}'$ be the subset of $\mathcal{L}$ for which the theorem is true.

Let $\tilde{\mathcal{L}} \subset \mathcal{L}'$ be a subset of $\mathcal{L}$ linearly ordered by set inclusion,

claim $\tilde{\cup} \tilde{\mathcal{L}}$ is in $\mathcal{L}'$, so is ~~a maximal element~~ an upper bound of $\tilde{\mathcal{L}}$.

Need a chain of Hamel bases $(H_\ell)_{\ell \in \tilde{\mathcal{L}}}$ s.t.

$H_{\ell_1} \subset H_{\ell_2}$ if $\ell_1 \subset \ell_2$.

Thm

$\mathcal{L}$ chain of linear spaces ordered by set inclusion.

$\exists$, chain of Hamel bases $(H_L)_{L \in \mathcal{L}}$ s.t.

$H_{L_1} \subseteq H_{L_2}$ if $L_1 \subseteq L_2$

Pf

Transfinite induction.

← [insert here.]

Let H = union of collection of Hamel bases $(H_\ell)_{\ell \in \tilde{\mathcal{L}}}$

s.t. $H_{\ell_1} \subseteq H_{\ell_2}$ if $\ell_1 \subseteq \ell_2$

Need ~~to~~ consider the towers of ~~all~~ chains of ~~linear~~ ~~Hamel bases~~ ~~for which~~ linear spaces which have the property that