Spherical Calculus

Let $\alpha(n)$ = volume of unit ball in $\mathbb{R}^n$.

Let $B(x,r)$ = ball centered at x of radius r.

Write $B_r(x) = B(x,r)$, $B_r = B_r(0)$.

Let $\fint_V u := \dfrac{\int_V u}{\mathrm{vol}(V)}$ $\quad \forall u, V$.

Let $\hat{z}(x;y)$ = unit normal = $\dfrac{y-x}{|y-x|}$

Then

(1) $$\int_{B(x,r)} u = \int_{\rho=0}^{r} \int_{\partial B(x,\rho)} u$$

(2) $$d_r \int_{B(x,r)} u = \int_{\partial B(x,r)} u$$

(3) $$\int_{B(x,r)} 1 = \alpha(n)\, r^n = \mathrm{vol}(B(x,r))$$

(4) $$\int_{\partial B(x,r)} 1 = n\alpha(n)\, r^{n-1} = \mathrm{vol}(\partial B(x,r))$$

(5) $$\fint_{B(x,r)} u = \frac{1}{\alpha(n) r^n} \int_{B(x,r)} u$$

(6) $$\fint_{\partial B(x,r)} u = \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} u$$

(7) $$\fint_{\partial B(x,r)} \hat{z}\cdot u = \frac{r}{n} \fint_{B(x,r)} \nabla\cdot u$$

(8*) $$d_r \fint_{\partial B(x,r)} u = \fint_{\partial B(x,r)} \hat{z}\cdot \nabla u = \frac{r}{n} \fint_{B(x,r)} \Delta u$$

(9*) $$d_r \fint_{B(x,r)} u = \frac{n}{r}\left(\fint_{\partial B(x,r)} u - \fint_{B(x,r)} u \right)$$

Proofs

(Pf (8*))

$$d_r \fint_{\partial B(x,r)} u = d_r \fint_{\partial B(0,1)} u(x+rz)\, dS(z)$$

$$= \fint_{\partial B(0,1)} Du(x+rz)\cdot z \, dS(z)$$

$$= \left[\begin{array}{l} \text{Let } y = x+rz, \\ \text{So } z = \frac{y-x}{r} = \hat{z} \end{array} \right.$$

$$= \fint_{\partial B(x,r)} Du(y)\cdot \hat{z}$$

$$\left(= \frac{r}{n} \fint \Delta u \quad \text{by (7)} \right)$$

(Pf 9*)

$$d_r \fint_{B(x,r)} u = d_r \left(\frac{1}{\alpha(n) r^n} \int_{B(x,r)} u \right)$$

$$= \frac{1}{\alpha(n)} \frac{-n}{r\cdot r^n} \int_{B(x,r)} u + \frac{1}{\alpha(n) r^n} \int_{\partial B(x,r)} u$$

$$= \frac{-n}{r} \fint_{B(x,r)} u + \frac{n}{r} \fint_{\partial B(x,r)} u$$

$$= \frac{n}{r}\left(\fint_{\partial B(x,r)} u - \fint_{B(x,r)} u \right)$$

Take $r \in \mathbb{R}^n$.

Find $\nabla |r|^m$.

$$[\nabla |r|^m]_i = \partial_i |r|^m$$
$$= \frac{d|r|^m}{d|r|} \partial_i |r|$$
$$= m|r|^{m-1} \partial_i \sqrt{r \cdot r}$$

So $\nabla |r|^m$

$$\partial_i \sqrt{r\cdot r} = \frac{d\sqrt{r\cdot r}}{d(r\cdot r)} \partial_i (r\cdot r)$$
$$= \frac{1}{2}(r\cdot r)^{-\frac{1}{2}} \cdot 2r_i$$
$$= \frac{r_i}{\sqrt{r\cdot r}} = \frac{r_i}{|r|}$$

So $[\nabla |r|^m]_i = m|r|^{m-2} r_i$

and $\nabla |r|^m = m|r|^{m-2} r$

$$\boxed{\nabla |r|^m = m|r|^{m-1} \hat{r}}$$

Find $\nabla^2 |r|^m = \nabla \cdot (\nabla |r|^m)$
$$= \nabla \cdot m|r|^{m-2} r$$
$$= \sum_{j=1}^n \partial_j (m|r|^{m-2} r_j)$$
$$= m\left(\sum_j \begin{bmatrix} (m-2)|r|^{m-4} r_j^2 \\ + |r|^{m-2} \end{bmatrix} \right)$$
$$= m\left((m-2)|r|^{m-2} + n|r|^{m-2} \right)$$
$$= m(m-2+n)|r|^{m-2}$$

Want $\nabla^2 |r|^m = 0$

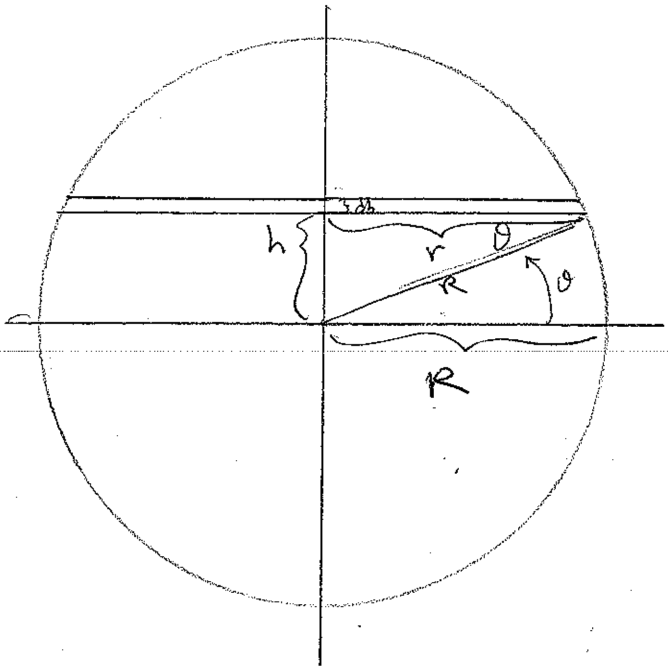
Need $\boxed{m = 2-n}$

So $\nabla^2 |r|^{2-n} = 0$

Want $\nabla^2 |r|^{2-n} = c\,\delta(r)$.

$$\int_{|r|\le R} \nabla^2 |r|^{2-n} dr$$
$$= \int_{|r|=R} \hat{n} \cdot \nabla |r|^{2-n} dr$$
$$= \int_{|r|=R} \hat{n} \cdot (2-n)|r|^{1-n} \hat{r}\, dr$$
$$= (2-n) R^{1-n} \cdot \text{Area (unit ball)}\, R^{n-1}$$
$$= (2-n) \cdot \text{Area (unit ball)}$$

If $n=3$
$$= -4\pi$$



Let $A_n(r)$ = volume of n-dim'l sphere of radius r.

$$A_{n+1}(R) = 2\int_{h=0}^{R} dh\, A_n(r)$$

$$= 2\int_{h=0}^{R} A_n(\sqrt{R^2-h^2})\, dh$$

$$= 2\int_{\theta=0}^{\frac{\pi}{2}} A_n(R\cos\theta)\, R\cos\theta\, d\theta.$$

$$A_1(r) = 2r$$

In general $A_n(r) = k_n r^n$

Want

$$k_{n+1} = 2\int_{h=0}^{1} A_n(r)\, dh$$

$$= 2\int_{h=0}^{1} k_n r^n\, dh$$

$$= 2\int_{0}^{\frac{\pi}{2}} k_n \cos^n\theta \cos\theta\, d\theta$$

$$h^2 + r^2 = R^2$$

$$r = \sqrt{R^2 - h^2}$$

$$= R\cos\theta$$

$$h = R\sin\theta$$

$$dh = R\cos\theta\, d\theta$$

$$r^n = R^n \cos^n\theta$$

Need $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n+1}\theta\, d\theta$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos^n\theta \sin\theta)' + n\cos^{n-1}\theta \sin^2\theta\, d\theta$$

Assume $n \geq 1$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} n[\cos^{n-1}\theta - \cos^{n+1}\theta]\, d\theta$$

So $\underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n+1}\theta\, d\theta}_{\text{Call } C_{n+1}} = \frac{n}{n+1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{n-1}\theta\, d\theta$

So $C_{n+1} = \frac{n}{n+1} C_{n-1}$

$$C_2 = \frac{1}{2} C_0$$

$$C_0 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \pi$$

$C_2 = \frac{\pi}{2}$, $C_4 = \frac{3}{4}\cdot\frac{\pi}{2}$, $\boxed{C_6 = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{\pi}{2}}$ $2n=6$, $n=3$

$$C_1 = \int_{-\frac{\pi}{2}}^{\pi/2} \cos\theta\, d\theta = 2.$$

$$C_3 = \frac{2}{3}\cdot 2 = \frac{4}{3}$$

$$C_5 = \frac{4}{5}\cdot\frac{2}{3}\cdot 2 =$$

$C_7 = \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cdot 2$ $2n+1 = 7$, $n = 3$

$$C_{2n+1} = \frac{(2^n n!)^2 \cdot 2}{(2n+1)!}$$

$$C_{2n} = \frac{(2n-1)!^2}{(2n)!}$$