Computational mechanics: what is it, why do I like it, and how can you learn it?

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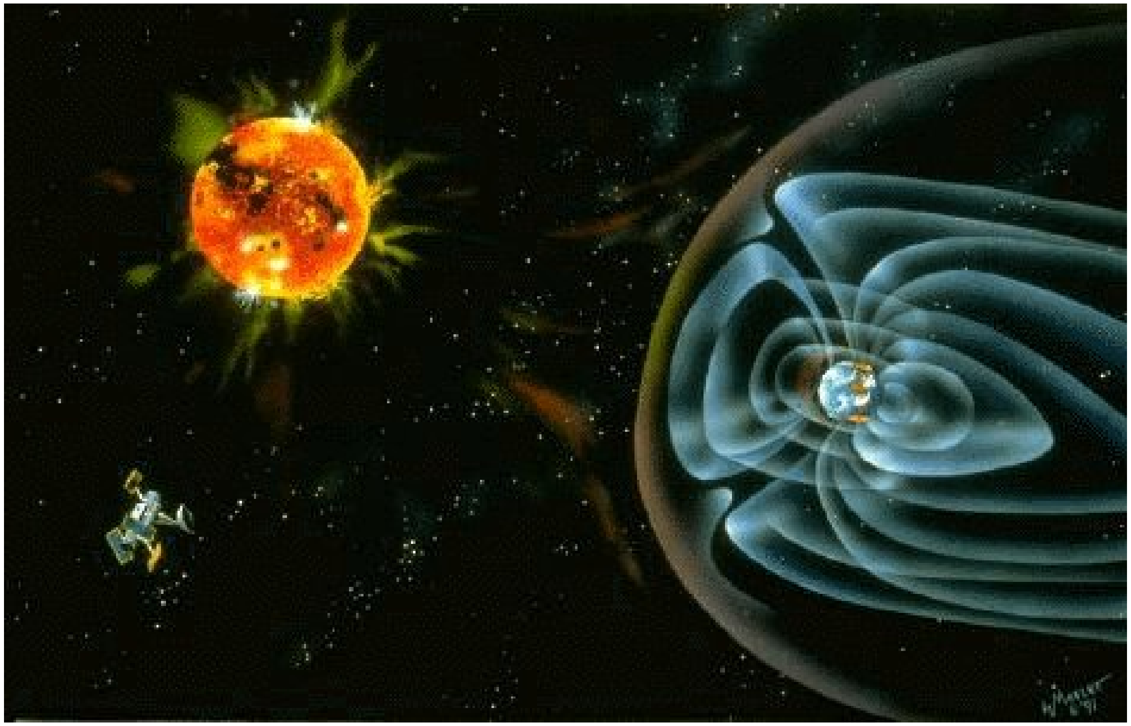
Presented on December 13, 2007
for high school math night



Outline

① My path (how I got to what I'm doing)

② My passion

(a) applied math
(b) computational mechanics
(c) plasmas

③ Problems

(a) modeling traffic flow
(b) computing gyroradius

Your journey: finding your way.

How do you navigate the academic world fruitfully?

① Cultivate disciplined habits.

- Have a sense of wonder.
- Be curious and observant.
- Take the long view.

② Look for good mentors.

- Talk to people who are doing things that interest you.
  - What is their passion?
  - How did they get where they are?

My story: where I am now.

- I'm a happy grad student!

① I love my advisor.
  (a) He's generous with his time.
  (b) He interacts well with me.
    i. He's good at what he does.
    ii. He's transparent about the limits of what he knows.

② I'm passionate about what I'm studying.
  (a) Plasmas are interesting.
  (b) There is a lot of work to do:
  There are problems that
    i. people care about,
    ii. are accessible to a graduate student, and
    iii. haven't been solved already.

My story: how I got here.

- I'm an older student (almost 33).
  - focused: entered grad school with a sense of purpose and mission
  - invested: my life was already invested with a certain course

- My previous adult activities
  - undergraduate geophysics research (ice and climate)
  - worked on a large software project ('98-01)
  - taught high school (math, physics, Greek) ('97-98, '01-02')
  - **Budapest Semesters in Mathematics** ('03)

My story: how I got here.

How I decided on math for grad school.

- I'm interested in everything fundamental
  - math, physics, computers,
  - Bible, historical linguistics, Christian origins
- math: center of my interests

My story: how I got here.

How I decided to study plasma physics.

- mechanics: I wanted to understand basic physics
- plasmas: combines many areas of physics
  (fluids, electromagnetism, relativity)
- computation: wanted to be able to say something concrete about the physical world

My passion.

① applied math

② continuum mechanics

③ plasma physics

My area: applied math.

What is applied math?

- happy medium between math and engineering
- traditionally: focus on fluid mechanics
- recently: shifting towards
  - multiscale modeling
  - biological systems
  - dynamics of information
  - cryptography

My skill: computational continuum mechanics

What is continuum mechanics?

- Discrete models view matter and energy as interacting discrete particles.
- Continuum models view stuff as continuous (as opposed to a collection of particles).
- Continuum mechanics is basically the study of **how stuff flows**.

- Continuum mechanists study equations that look like:

$$\frac{\partial q}{\partial t} + \frac{\partial f}{\partial x} = s$$

  - $q = q(x, t)$ = density of stuff
  - $f = f(x, t)$ = flux (flow rate of the stuff)
  - $s(x, t)$ = source/sink = rate at which stuff is produced/destroyed

My topic: plasma physics

- What is a Plasma?
  - As matter is heated, it disassociates:
    - solid (rigid bonds)
    - liquid (molecules held together)
    - gas (molecules free)
    - plasma (free electrons and ions)
  - So plasma $=$ an electrically charged fluid

- examples of plasma
  - lightning
  - polar auroras
  - sun
  - **solar wind**
  - fluorescent lights
  - $> 99\%$ of matter in the universe

Problem: space weather

My motivational focus: model space weather

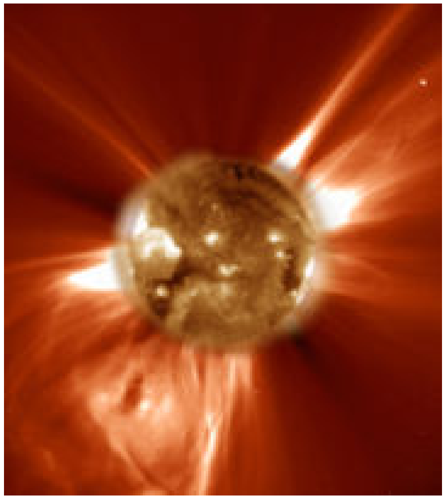
① What is space weather?

② Why is it important?

Problem: space weather

What is **space weather**?

- Earth bombarded with **solar wind**.
  1. *charged*: mostly protons or electrons.
  2. *sparse*: 5-10 protons (or electrons) per cm^3.
  3. *fast-moving*: proton velocities of 200–800 km/s.
- Solar wind varies dramatically.
- Solar storms cause **geomagnetic storms**.



Coronal Mass Ejection.

Problem: Why model space weather?

Why are people interested in modeling space weather?

- Geomagnetic storms affect us.
  - Produce polar auroras.
  - Damage power grids and satellites
  - Disrupt communications
- Space weather is hard to model because plasmas can be violent and erratic
  - Sun is unpredictable
  - magnetic field lines can rapidly cancel in a phenomenon called *fast reconnection*, resulting in solar and geomagnetic storms.
  - what triggers fast reconnection is not well understood.

Model equations: Conservation law framework

We express our equations as conservation (balance) laws.

$$\underline{q}_t + \underline{\nabla} \cdot \underline{\underline{f}}(\underline{q}) = \underline{s}(\underline{q}).$$

- $\underline{q}$ is the state, i.e., a tuple of conserved "stuff",
- $\underline{\underline{f}}$ is the flux, i.e., the rate at which each type of stuff flows, and
- $\underline{s}$ is the source term, i.e., the rate at which each kind of stuff is produced or destroyed.

If we partition space into discrete cells and partition time into discrete time intervals this equation says that the change in the amount of stuff in a cell over one time step equals the net amount that flowed into the cell across the cell interfaces plus the net amount of stuff that was produced inside the cell.

$$\int_C \underline{q}(t_{n+1}) = \int_C \underline{q}(t_n) - \int_{t=t_n}^{t_{n+1}} \int_C \mathbf{n} \cdot \underline{\underline{f}} + \int_{t=t_n}^{t_{n+1}} \int_C \underline{s},$$

where C denotes a spatial cell and t_n and t_{n+1} denote successive time steps.

Model equations: Ideal MHD

Below are the equations for the simplest model of plasma, called *magnetohydrodynamics*.

The full system of ideal MHD equations is

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{u}\\ \tilde{\mathcal{E}}\\ \mathbf{B}\end{bmatrix} + \nabla\cdot\underbrace{\begin{bmatrix}\rho\mathbf{u}\\ \rho\mathbf{u}\mathbf{u}+\tilde{p}\,\mathbb{I}-\frac{1}{\mu_0}\mathbf{B}\mathbf{B}\\ \mathbf{u}\big(\tilde{\mathcal{E}}+\tilde{p}\big)-\frac{1}{\mu_0}\mathbf{B}\mathbf{B}\cdot\mathbf{u}\\ \mathbf{u}\mathbf{B}-\mathbf{B}\mathbf{u}\end{bmatrix}}_{\text{flux}} = 0$$

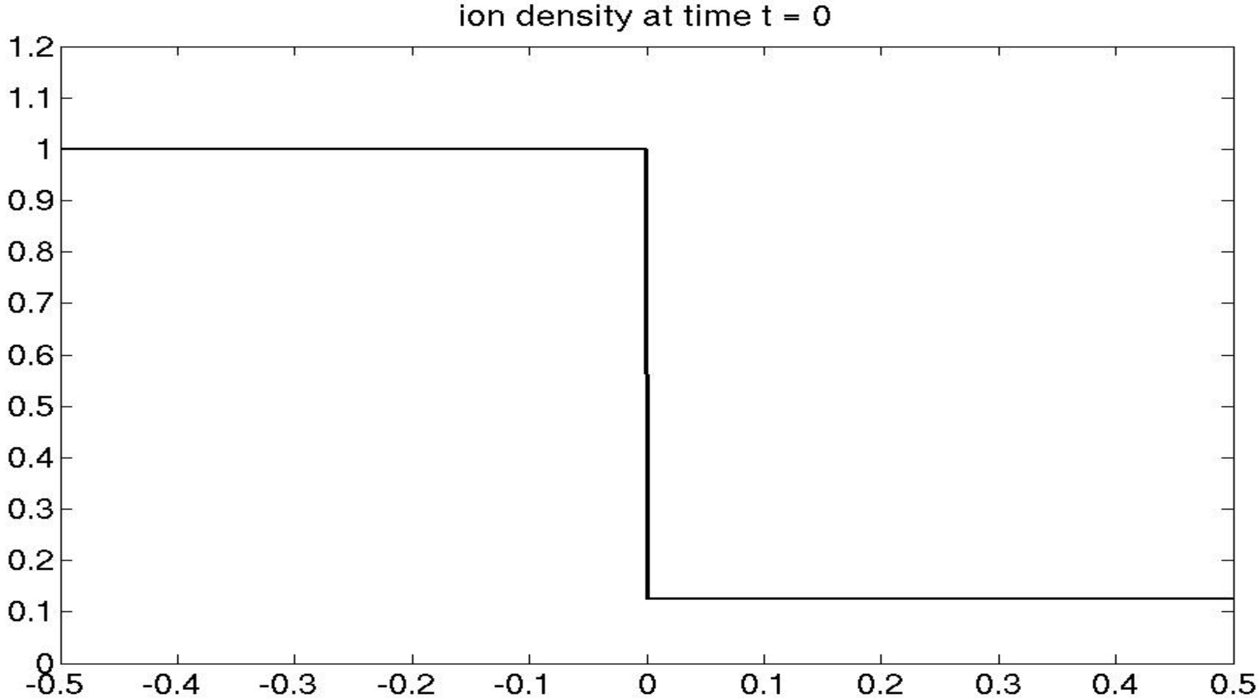
Numerical Method: Choice of problem.

In order to simulate plasma you have to set **initial conditions** that determine its state at time zero.

We set initial conditions to the left and right of zero as:

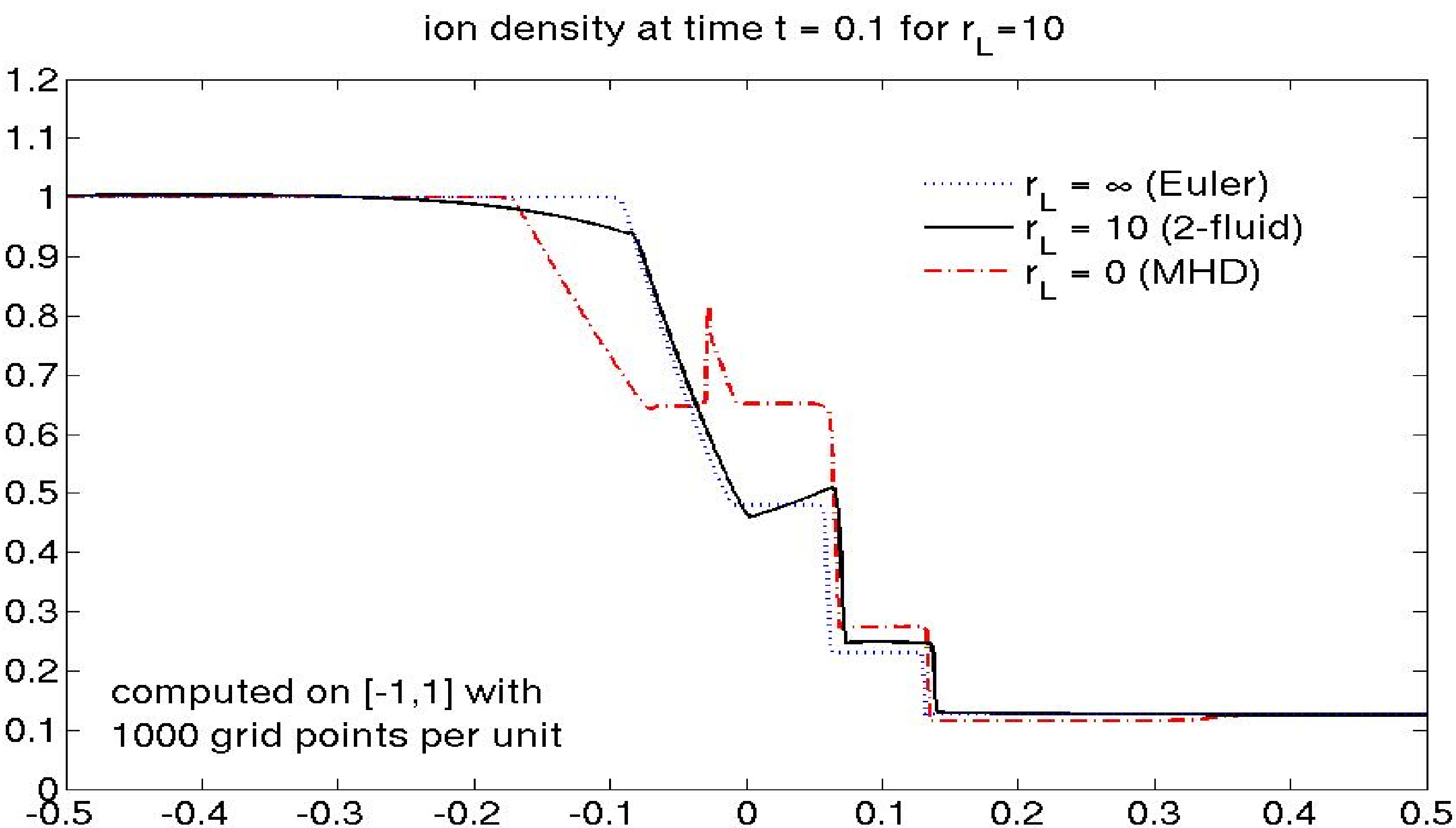
$$\begin{bmatrix} \rho \\ v^1 \\ v^2 \\ v^3 \\ p \\ B^1 \\ B^2 \\ B^3 \end{bmatrix}_{\text{left}} = \begin{bmatrix} 1.0 \\ 0 \\ 0 \\ 0 \\ 1.0 \\ 0.75 \\ 1.0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \rho \\ v^1 \\ v^2 \\ v^3 \\ p \\ B^1 \\ B^2 \\ B^3 \end{bmatrix}_{\text{right}} = \begin{bmatrix} 0.125 \\ 0 \\ 0 \\ 0 \\ 0.1 \\ 0.75 \\ -1.0 \\ 0 \end{bmatrix}$$

Numerical Method: Results.



The initial ion density is piecewise constant with a single discontinuity at zero.

Numerical Method: Results.



After .1 second the shock (dotted red line) has spread out. (Other plots are for comparison.)

Acknowledgements.

I wish to thank:

- Wisconsin Space Grant Consortium (`http://www.uwgb.edu/WSGC/`)
- James Rossmanith (my advisor)